Univ. of [illegible]

THE PHYSICAL REVIEW

A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

CONDUCTED BY

THE AMERICAN PHYSICAL SOCIETY

BOARD OF EDITORS

FREDERICK BEDELL, Managing Editor

HENRY CREW — E. L. NICHOLS — G. S. FULCHER
L. V. KING — C. M. SPARROW — F. A. SAUNDERS
H. S. UHLER — W. F. G. SWANN — O. M. STEWART

FEBRUARY, 1921

The Absolute Significance of Maxwell's Equations — F. D. MURNAGHAN 73
The Formation of Negative Ions in Air — LEONARD B. LOEB 89
Theory and Calculation of Variable Electrical Systems — JOHN R. CARSON 118
The Blackening of a Photographic Plate as a Function of Intensity of Light and Time of Exposure — P. S. HELLICK 135
Some Applications of the Method of Images. I — JACOB KUNZ and P. L. BAYLEY 149
Some Applications of the Method of Images. II — C. M. HERBERT 157
The Electrical Resistance of Metals — P. W. BRIDGMAN 161
The Luminosity of Mercury Vapor Distilled from the Arc in Vacuo — NORMAN H. KICKER 183
[illegible] 197

Henry A. Rumstead

(Contents continued on back cover page)

PUBLISHED BY
THE AMERICAN PHYSICAL SOCIETY
LANCASTER, PA., AND ITHACA, N. Y.
Entered at the postoffice at Lancaster, Pa., as second class matter

(CONTENTS—Continued)

Proceedings of the American Physical Society 225

Minutes of the One hundred fifth Meeting; The Crystalline Structure of Sputtered Metallic Films, *H. Kahler;* The Exact of Adsorbed Air on the Aging of Thin Metal Films, *Lewis R. Koller;* Constancy of Hall Coefficient in Thin Silver Films, *G. W. Stewart;* The Hall Effect and the Specific Resistance in Silver Films, *G. R. Wait;* The Hall Effect and the Nernst Effect in Magnetic Alloys, *Alpheus W. Smith;* Conductivity of Insulating Materials near the Breakdown Voltage, *James E. Snyder;* Magneto-Resistance Effects in Films of Bismuth, *L. F. Curtiss;* The Magnetic Mechanical Analysis of Manganese Steel, *Robert Hadfield, S. R. Williams and I. S. Bowen;* Pulsating Thermionic Discharges in Evacuated Tungsten Lamps, *A. G. Worthing;* Effect of Strong Electrostatic Fields on the Vaporization of Tungsten, *A. G. Worthing and W. C. Baker;* A Compensation Scheme for Electrostatic Measurements, *W. F. G. Swann;* Note on Electromagnetic Induction and Relative Motion. A Rejoinder, *W. F. G. Swann;* The Electrostatic Charges of the Earth and Sun and their Relation to Terrestrial Magnetism, *Fernando Sanford;* The Spectrophotoelectric Sensitivity of Proustite, *W. W. Coblentz;* Color Filters for Photographic Uses, *Charles D. Hodgman;* Comparison between the Fundamental Equations for the Pondero-motive Force for Point Charges Due to Larmor-Lorentz and Mega Nad Saha, *Albert C. Crehore;* The Newtonian Laws of Gravitation Deduced from the Saha Electromagnetic Theory Applied to the Copernican Atom, *Albert C. Crehore;* On the Propagation of the Sound Wave from the Muzzle of a Large Gun, *Dayton C. Miller;* A New Tone Generator, *C. W. Hewlett;* A Method of Studying Sound Waves by Means of a Synchronous Commutator, *L. O. Grondahl;* On the Relative Positions of Lines in X-Ray Spectra, *William Duane and R. A. Patterson;* The Relation between the Emissive Power of a Metal and its Electrical Resistivity, *C. Davisson and J. R. Weeks;* Mass-Absorption Coefficients as a Function of Wave-Length above and below the K X-Ray Limit of the Absorber, *F. K. Richtmyer;* Graphical Determination of Hexagonal and Tetragonal Crystal Structure from X-Ray Data, *Albert W. Hull and Wheeler P. Davey;* The Absorption and Scattering Coefficients for Homogeneous X-Rays in Several Elements of Low Atomic Weight, *C. W. Hewlett;* The Current-Temperature Relation for Different Pyrometer Filaments, *W. E. Forsythe;* On the Absorption Spectrum of HCl, *Walter F. Colby and Charles F. Meyer;* Flame Excitation of Luminescence, *E. L. Nichols and D. T. Wilber;* A Continuous Spectrum from Mercury Vapor, *Clement D. Child;* Note on Sheet Strias as Thin Films in Optical Glass, *E. L. Dodd.*

New Books 272

Second Series *February, 1921* *Vol. XVII, No. 2*

THE

PHYSICAL REVIEW.

THE ABSOLUTE SIGNIFICANCE OF MAXWELL'S EQUATIONS.

By F. D. Murnaghan.

Synopsis.

Elementary Proof of a Generalized Stokes' Theorem for Hyper-space.—After giving precise definitions of the ideas of integrals extended over lines, surfaces and hyper-surfaces in a space of arbitrary metrical character, the author presents an elementary proof of a generalized Stokes' theorem which reduces the integral over an open hyper-surface to an integral over the closed surfaces which bound it.

Statement of Maxwell's Equations as Integrals over Hyper-surfaces, brings out their physical significance more clearly as relations between surface integrals and hyper-surface integrals. It is explained how this form of statement lends itself at once to the methods of the absolute differential calculus.

Electromagnetic Potential.—The *existence* of this potential, which is a hyper-space combination of vector and scalar potentials, is proved and the normal form of its *line integral over any open curve* is given (1) for an electromagnetic field where $(\overline{E}.\overline{B}) = 0$, and (2) for any other field.

THE fundamental importance of Maxwell's electromagnetic equations has long been recognized. In recent years, however, attention has been directed to the *absolute* character of physical equations and the aim has been to express physical laws in a manner independent of the particular coördinates used to give measures of the quantities involved. It may be well, then, to point out explicitly the absolute content of Maxwell's equations, or in other words to answer the query "what absolute relation connecting the vectors of electromagnetism with space and time is expressed by these fundamental equations?" To explain further what is attempted here let us recall the *induction vector* $\overline{B}$ which is so important to the electrical engineer; to tell a student of engineering that the induction vector $\overline{B}$ is more important or fundamental than the magnetic force vector $\overline{H}$, for example, because its divergence $(\partial B_x/\partial x) + (\partial B_y/\partial y) + (\partial B_z/\partial z)$ is zero is not so satisfactory or illuminating as to

express the same fact by saying that the surface integral $\int B_n dS$ over any closed surface is zero. This absolute statement gives at once a useful picture of what is taking place and leads to the conception of induction lines and tubes which is so fruitful. Now it so happens that the content of Maxwell's equations can be stated in an analogous absolute form and we can apply to them the methods of the Absolute Calculus explained in Einstein's paper on General Relativity in the Annalen der Physik (1916). However we shall have to speak of four dimensional space and as it is desirable to allow gravitational, *i.e.*, non-euclidean spaces it may be permitted to recall the mathematical definition of space and to show that the results obtained in no way depend on the metrical properties of the space.[1]

§ 1.

For definiteness we shall speak of a four-dimensional space although the argument is identical for a space of any number of dimensions. By the term *point* is meant an ordered set of four numbers (x_1, x_2, x_3, x_4), which set is for brevity spoken of as the point x—the numbers being called the four coördinates of the point. By the term space we mean simply the aggregate of all the points x but we shall suppose that our space is real and continuous *i.e.*, the numbers (x_1, x_2, x_3, x_4) can each take up all real values in a continuous manner. If the coördinates of a variable point x are expressible as continuous functions of a single independent variable u, say,

$$x_i = x_i(u), \qquad i = 1, 2, 3, 4,$$

the point x is said to trace a *curve* in the space, as the independent variable u varies; we shall suppose that our curves are *smooth i.e.*, the functions $x_i(u)$ have continuous derivatives with respect to u.[2] But if the coördinates of a variable point x have more freedom and are expressible as functions of two independent variables (u_1, u_2), say,

$$x_i = x_i(u_1, u_2), \qquad i = 1, 2, 3, 4,$$

the point is said to trace out a surface or variety of two dimensions as the independent variables (u_1, u_2) vary. Again we make the hypothesis of smoothness. The functions $x_i(u_1, u_2)$ must be *distinct, i.e.*, not all

[1] Most of the results stated in this note are familiar to several writers on Mathematical Physics and the reader is referred to a paper by Bateman, Proc. Lond. Math. Soc., Ser. 2, Vol. (8), 1910, p. 223, for further developments. It is hoped that the elementary presentation here given will enable physicists to become more easily familiar with the important results of this and similar papers.

[2] It will not affect our argument if any of our curves have several "corners" *i.e.*, the derivatives may be discontinuous at a finite number of values of u.

the Jacobian determinants

$$\frac{\partial(x_i x_j)}{\partial(u_1 u_2)} \equiv \begin{vmatrix} \frac{\partial x_i}{\partial u_1} & \frac{\partial x_i}{\partial u_2} \\ \frac{\partial x_j}{\partial u_1} & \frac{\partial x_j}{\partial u_2} \end{vmatrix} \qquad i, j = 1, 2, 3, 4,$$

can vanish for all values of u_1, u_2. If this were the case our point x would not "have more freedom" and would really trace a curve. Similarly if $x_i = x_i(u_1, u_2, u_3)$, where the x_i are distinct functions of three independent variables (u_1, u_2, u_3), the point is said to trace a hypersurface or variety of three dimensions as the independent variables vary. Once more we make the hypothesis of smoothness of our variety. It remains to define the terms line integral, surface integral, hypersurface integral and as we are not using any metrical properties of space we shall avoid terms such as "element of arc," "surface element," "volume element" so familiar in this connection.

Line Integral.

By

$$\int_C (A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4) \equiv \int_C \sum_{s=1}^{4} (A_s dx_s)$$

over a curve C whose equations are $x_i = x_i(u)$, $i = 1, 2, 3, 4$, and where the A's are given functions of the variables (x_1, x_2, x_3, x_4), we mean that we write in for each x_i its value $x_i(u)$ and for dx_i its value $(dx_i/du)du$ and then evaluate the definite integral

$$\int_{u^0}^{u'} \left\{ \sum_s \left(A_s \frac{dx_s}{du} \right) \right\} du$$

u^0 and u' being the values of the independent variable, or parameter, u corresponding to the end points of the curve C. It is important to notice that this line integral is really *attached to the curve, i.e.*, its value is independent of the particular parameter u used to define the curve; for suppose we change the parameter u to another v by an equation $u = u(v)$, our line integral is

$$\int_{v^0}^{v'} \left\{ \sum_s A_s \frac{dx_s}{dv} \right\} dv$$

and by the well-known rule for changing the variable of integration the previous value is

$$\int_{v^0}^{v'} \left(\sum A_s \frac{dx_s}{du} \right) \frac{du}{dv} dv,$$

which is identical with that just written since

$$\frac{dx_s}{dv} = \frac{dx_s}{du} \cdot \frac{du}{dv}.$$

Surface Integral.

Consider a set $A_{s_1s_2}$, $s_1, s_2 = 1, 2, 3, 4$, of functions of (x_1, x_2, x_3, x_4); since s_1, s_2 may each take four values there are 16 functions; but we shall make the hypothesis that the $A_{s_1s_2}$ form a skew symmetric set, *i.e.*, a permutation of the suffixes s_1, s_2 merely alters the sign of the function. This makes the four functions $A_{11}, \cdots, A_{44}$ identically zero and the remaining twelve are, in pairs, equal in magnitude but opposite in sign. There are, then, really only six essentially distinct functions $A_{s_1s_2}$. Now when we are given any surface

$$x_i = x_i(u_1, u_2), \qquad i = 1, 2, 3, 4,$$

we may write down the double integral

$$I_2 \equiv \int \left\{ \sum_{s_1s_2}^{1\ldots4} A_{s_1s_2} \frac{\partial x_{s_1}}{\partial u_1} \frac{\partial x_{s_2}}{\partial u_2} \right\} du_1 du_2$$

extended over that range of values of (u_1, u_2) which corresponds to the surface [each x being replaced by $x(u_1, u_2)$]. The double summation under the integral sign is extended over all permutations of s_1, s_2 but we can, on account of the skew-symmetric relations $A_{s_1s_2} = -A_{s_2s_1}$, group the terms where the same combination s_1s_2 occurs and we obtain

$$I_2 = \int \left\{ \sum_{s_1s_2}{}' A_{s_1s_2} \frac{\partial(x_{s_1}, x_{s_2})}{\partial(u_1, u_2)} \right\} du_1 du_2$$

where the prime attached to the sign of summation indicates that $s_1 < s_2$. As before we may satisfy ourselves that this integral is really *attached to the surface* and in no way depends for its value on the accidental choice of parameters (u_1, u_2) used to specify the surface. For if parameters (v_1, v_2) are used we have as equations connecting the parameter pairs $u_1 = u_1(v_1, v_2)$; $u_2 = u_2(v_1, v_2)$ and the well-known formula for change of variables in a double integral gives us

$$I_2 = \int \left\{ \sum_{s_1s_2}{}' A_{s_1s_2} \frac{\partial(x_{s_1}, x_{s_2})}{\partial(u_1, u_2)} \right\} \frac{\partial(u_1, u_2)}{\partial(v_1, v_2)} dv_1 dv_2$$

—the integral being extended over the region of values (v_1, v_2) corresponding to the previous region of values (u_1, u_2)—and this by the elementary theorem as to the multiplication of Jacobians is

$$\int \left\{ \sum_{s_1s_2}{}' A_{s_1s_2} \frac{\partial(x_{s_1}, x_{s_2})}{\partial(v_1, v_2)} \right\} dv_1 dv_2.$$

In order to bring into prominence this independence of the choice of parameter I_2 is usually written in the abbreviated form $\int \sum' A_{s_1s_2} dx_{s_1} dx_{s_2}$ which form is, then, merely a *convenient way of writing* the surface integral defined in this paragraph.

Hyper-Surface Integral.

After the previous details this can be explained briefly. Consider the set of $4^3 = 64$ functions $A_{s_1 s_2 s_3} \cdots s_1, s_2, s_3 = 1, 2, 3, 4$, and agree that a mere interchange of two suffixes simply changes the sign of the function. Forty of the functions become identically zero and the remaining 24 arrange themselves in four groups of six functions, each group consisting of the same function three times with a positive and three times with a negative sign. We define

$$I_3 \equiv \int \left\{ \sum_{s_1 s_2 s_3}^{1 \cdots 4} A_{s_1 s_2 s_3} \frac{\partial x_{s_1}}{\partial u_1} \frac{\partial x_{s_2}}{\partial u_2} \frac{\partial x_{s_3}}{\partial u_3} \right\} du_1 du_2 du_3,$$

where each x_i is replaced by $x_i(u_1, u_2, u_3)$, $i = 1, 2, 3, 4$, from the equation of the hypersurface and the triple integral is extended over the range of values (u_1, u_2, u_3) corresponding to the hypersurface. Grouping the terms as before we have

$$I_3 = \int \left\{ \sum_{s_1 s_2 s_3}' A_{s_1 s_2 s_3} \frac{\partial(x_{s_1}, x_{s_2}, x_{s_3})}{\partial(u_1, u_2, u_3)} \right\} du_1 du_2 du_3,$$

where the prime indicates that $s_1 < s_2 < s_3$. This form shows that I_3 is really attached to the hypersurface and to indicate this we abbreviate it to

$$I_3 = \int \sum_{s_1 s_2 s_3}' A_{s_1 s_2 s_3} dx_{s_1} dx_{s_2} dx_{s_3}$$

there being but four terms in the summation.

By the term closed variety, *i.e.*, curve, surface, hypersurface we mean one which has no boundaries. (By the boundary of a curve we mean its endpoints; of a surface its bounding curve and of a hypersurface its bounding surface.) A variety which has boundaries is said to be open and it is clear[1] that any open variety lying at a finite distance (*i.e.*, all of whose points have finite coördinates) has a *closed* variety of one less dimension as its boundary. We have not defined points with infinite coördinates so we shall not discuss them here. There is an important relation known as Stokes' generalized lemma connecting an integral over any open variety with an integral over its closed boundary. It will be seen that this yields as special cases those fundamental theorems of Mathematical Physics known as Stokes' Theorem, Green's Lemma, Gauss' Theorem. From a pedagogical point of view it seems preferable to discuss these important theorems in this connected way and not to give more or less complicated proofs of each theorem separately; proofs which, moreover, give no hint of the close relation between the theorems.

[1] Bearing in mind that the coördinates of points on any variety are *continuous* functions of the parameters used in specifying the variety.

It will be best to start with the simplest case although the method is the same for all.

Case 1. *Open Surface Bounded by One or More Closed Curves.*

There is no lack of generality in supposing that there is but one closed curve for boundary for we can connect all the separate curves by lines drawn on the surface in such a way that in tracing the boundary these lines will each be traversed twice—in opposite directions. Hence any integral extended over the new boundary will be the sum of the values of the same integral extended over the original closed curves—the contributions from the new connecting lines cancelling out. The boundary will be given by a relation between the parameters $\phi(u_1, u_2) = 0$. It will be convenient to avail ourselves of the arbitrariness as to the choice of parameters and we write $v_2 = \phi(u_1, u_2)$, $v_1 = f(u_1, u_2)$, where f is any convenient function distinct from ϕ. The advantage of the new parameters is that the boundary is given by $v_2 = 0$ and so is one of the curves $v_2 = \text{const.}$ All these curves are closed, since they cannot intersect the boundary;[1] and we obtain the entire surface as traced out by the parameter curves $v_2 = \text{constant}$, the constant varying for each curve from zero to some fixed value. Consider now a line integral

$$I_1 = \int \sum_{s_1}^{1\ldots4} (A_{s_1} dx_{s_1})$$

attached to one of these level curves; v_2 is to be constant in the integration and I_1 is a function of v_2 which merely says that I_1 will depend on the particular curve v_2 along which the integral is extended. Writing out I_1 in full,

$$I_1 = \int \left\{ \sum_{s_1}^{1\ldots4} A_{s_1} \frac{\partial x_{s_1}}{\partial v_1} \right\} dv_1,$$

we get on differentiating

$$\frac{dI_1}{dv_2} = \int \sum_{s_1}^{1\ldots4} \left(A_{s_1} \frac{\partial^2 x_{s_1}}{\partial v_1 \partial v_2} + \sum_{s_2}^{1\ldots4} \frac{\partial A_{s_1}}{\partial x_{s_2}} \frac{\partial x_{s_2}}{\partial v_2} \frac{\partial x_{s_1}}{\partial v_1} \right) dv_1,$$

where we have remembered that A_{s_1} is not directly a function of the parameters v but a function of the x's. Now along any *closed* curve

$$\int \frac{\partial F}{\partial v_1} dv_1 = 0,$$

where F is any *one-valued* function of position because the integral is simply the difference of the values of F at the end points of the curve.

[1] The range of values of the parameters of a surface is always supposed to be so restricted that the same point x on the surface cannot correspond to different values of the parameters.

Take

$$F = \sum_{s_1}^{1\ldots4} A_{s_1} \frac{\partial x_{s_1}}{\partial v_2}$$

and we have

$$\int \left\{ \sum_{s_1} \left(A_{s_1} \frac{\partial^2 x_{s_1}}{\partial v_1 \partial v_2} + \sum_{s_2}^{1\ldots4} \frac{\partial A_{s_1}}{\partial x_{s_2}} \frac{\partial x_{s_2}}{\partial v_1} \frac{\partial x_{s_1}}{\partial v_2} \right) \right\} dv_1 = 0.$$

From this we have

$$\frac{dI_1}{dv_2} = \int \left\{ \sum_{s_1, s_2}^{1\ldots4} \left(\frac{\partial A_{s_1}}{\partial x_{s_2}} - \frac{\partial A_{s_2}}{\partial x_{s_1}} \right) \frac{\partial x_{s_1}}{\partial v_1} \frac{\partial x_{s_2}}{\partial v_2} \right\} dv_1,$$

where we have interchanged in the latter term the symbols of summation s_1 and s_2. Writing now

$$A_{s_1 s_2} \equiv \frac{\partial A_{s_1}}{\partial x_{s_2}} - \frac{\partial A_{s_2}}{\partial x_{s_1}}$$

we see that it is skew symmetric and we obtain

$$I_1 = \int \sum{}' A_{s_1 s_2} dx_{s_1} dx_{s_2}$$

(there is no constant of integration to be added since the curves $v_2 =$ const. must close to a point[1] and so I_1 at one of the limits for v_2 is zero). (It might be argued that we should write $-I_1$ on the left-hand side of this last equation but there is an indeterminateness in the sign of $\int \sum{}' A_{s_1 s_2} dx_{s_1} dx_{s_2}$ anyhow. When we said that "the integral" was independent of the parameters it would have been more accurate to say its "absolute value." When the parameter transformation is direct the sign is unchanged, when inverse it is changed. The Jacobian of the transformation—$\partial(u_1 u_2)/\partial(v_1 v_2)$—is not supposed to vanish and so it is definitely positive or negative; when positive the transformation is direct, otherwise inverse. This change in sign is expressed geometrically by referring to the *two sides of a surface* and when we write

$$\int \sum_{s_1}^{1\ldots4} A_{s_1} dx_{s_1} = + \int \sum{}' A_{s_1 s_2} dx_{s_1} dx_{s_2}$$

the surface integral is said to be extended over that side of the surface which is positively related to the direction of integration along the boundary). Attention is called to the fact that when all $A_{s_1 s_2} \equiv 0$ the integral $I_1 = \int \sum A_{s_1} dx_{s_1}$ attached to *any closed curve* is zero. In this case I_1 is said to be the *integral of an exact differential.* The conditions $A_{s_1 s_2} \equiv 0$ are necessary as well as sufficient for this. To see this, take the surface $x_{s_1} = u_1$, $x_{s_2} = u_2$, whilst the other x's are constant; the surface integral reduces to $\int A_{s_1 s_2} du_1 du_2 = 0$ and this is to be zero for

[1] Or else a segment of a curve traced twice in opposite directions; for example the curves might be confocal ellipses.

any portion of the surface. If $A_{s_1 s_2} \not\equiv 0$ we take a point where it is positive and on account of continuity it will be definitely positive in the neighborhood of this point and the integral could not vanish over a small portion of the surface $x_{s_1} = u_1$; $x_{s_2} = u_2$ lying in this neighborhood.

Case 2. Open Hyper-surface Bounded by One or More Closed Surfaces.

Just as in Case 1 there is no lack of generality in supposing the boundary to consist of a *single* closed surface which will be given by a relation $\phi(u_1, u_2, u_3) = 0$ on the parameters (u_1, u_2, u_3) of the hypersurface. Make a change of parameters by writing v_1, v_2, v_3 as distinct functions of (u_1, u_2, u_3) and such that $v_3 \equiv \phi(u_1, u_2, u_3)$. Then the surfaces $v_3 =$ const. trace out the hypersurface as the constant varies; they are *closed* surfaces and the curves got on any one of them by putting $v_1 =$ const. or $v_2 =$ const. are *closed* curves. Consider now the integral

$$I_2 = \int \sum_{s_1 s_2}^{1 \ldots 4} A_{s_1 s_2} dx_{s_1} dx_{s_2}$$

attached to one of the surfaces $v_3 =$ const. On writing I_2 out explicitly and differentiating we have

$$\frac{dI_2}{dv_3} = \int \sum_{s_1 s_2}^{1 \ldots 4} A_{s_1 s_2} \left(\frac{\partial^2 x_{s_1}}{\partial v_1 \partial v_3} \frac{\partial x_{s_2}}{\partial v_2} + \frac{\partial x_{s_1}}{\partial v_1} \frac{\partial^2 x_{s_2}}{\partial v_2 \partial v_3} + \sum_{s_3} \frac{\partial A_{s_1 s_2}}{\partial x_{s_3}} \frac{\partial x_{s_1}}{\partial v_1} \frac{\partial x_{s_2}}{\partial v_2} \frac{\partial x_{s_3}}{\partial v_3} \right) dv_1 dv_2.$$

Now if F is a single-valued function of the x's the integral $\int (\partial F / \partial v_1) dv_1 dv_2$ extended over any of the surfaces $v_3 =$ const., is zero since the curves $v_2 =$ const. on that surface are closed. Take

$$F \equiv \sum_{s_1 s_2}^{1 \ldots 4} A_{s_1 s_2} \frac{\partial x_{s_1}}{\partial v_3} \frac{\partial x_{s_2}}{\partial v_2}$$

and we get

$$\int \sum_{s_1 s_2}^{1 \ldots 4} \left\{ A_{s_1 s_2} \left(\frac{\partial^2 x_{s_1}}{\partial v_1 \partial v_3} \frac{\partial x_{s_2}}{\partial v_2} + \frac{\partial x_{s_1}}{\partial v_3} \frac{\partial^2 x_{s_2}}{\partial v_2 \partial v_1} \right) + \sum_{s_3} \frac{\partial A_{s_1 s_2}}{\partial x_{s_3}} \frac{\partial x_{s_1}}{\partial v_3} \frac{\partial x_{s_2}}{\partial v_2} \frac{\partial x_{s_3}}{\partial v_1} \right\} dv_1 dv_2 = 0.$$

Similarly

$$\int \sum_{s_1 s_2}^{1 \ldots 4} \left\{ A_{s_1 s_2} \left(\frac{\partial^2 x_{s_1}}{\partial v_1 \partial v_2} \frac{\partial x_{s_2}}{\partial v_3} + \frac{\partial x_{s_1}}{\partial v_1} \frac{\partial^2 x_{s_2}}{\partial v_2 \partial v_3} \right) + \sum_{s_3} \frac{\partial A_{s_1 s_2}}{\partial x_{s_3}} \frac{\partial x_{s_1}}{\partial v_1} \frac{\partial x_{s_2}}{\partial v_3} \frac{\partial x_{s_3}}{\partial v_2} \right\} dv_1 dv_2 = 0.$$

Add together these two equations and notice that

$$\sum_{s_1 s_2}^{1\ldots 4} A_{s_1 s_2}\left(\frac{\partial x_{s_1}}{\partial v_3}\frac{\partial^2 x_{s_2}}{\partial v_2 \partial v_1}+\frac{\partial x_{s_2}}{\partial v_3}\frac{\partial^2 x_{s_1}}{\partial v_1 \partial v_2}\right)=0$$

because an interchange of the symbols $s_1 s_2$ of summation should not change its value whilst this interchange does in effect change its sign on account of the skew symmetric character of $A_{s_1 s_2}$. On making some obvious interchanges in the symbols of summation s_1, s_2, s_3 we accordingly obtain

$$\frac{dI_2}{dv_3}=\int \sum_{s_1 s_2 s_3}^{1\ldots 4}\left\{\left(\frac{\partial A_{s_1 s_2}}{\partial x_{s_3}}+\frac{\partial A_{s_2 s_3}}{\partial x_{s_1}}+\frac{\partial A_{s_3 s_1}}{\partial x_{s_2}}\right)\frac{\partial x_{s_1}}{\partial v_1}\frac{\partial x_{s_2}}{\partial v_2}\frac{\partial x_{s_3}}{\partial v_3}\right\} dv_1 dv_2.$$

Writing now

$$A_{s_1 s_2 s_3}\equiv\frac{\partial A_{s_1 s_2}}{\partial x_{s_3}}+\frac{\partial A_{s_2 s_3}}{\partial x_{s_1}}+\frac{\partial A_{s_3 s_1}}{\partial x_{s_2}}$$

we notice that it is skew-symmetric and we can write, on integrating with respect to v_3,

$$I_2=\int \sum_{s_1 s_2 s_3}' A_{s_1 s_2 s_3} dx_{s_1} dx_{s_2} dx_{s_3},$$

the integral being extended over the hypersurface. Attention is called to the fact that, if all $A_{s_1 s_2 s_3}\equiv 0$, I_2 extended over any *closed* surface is zero and in this case I_2 is said to be the *integral of an exact differential*. Just as in Case I the vanishing of all $A_{s_1 s_2 s_3}$ may be shown to be a necessary as well as a sufficient condition for

$$I_2\equiv\sum_{s_1 s_2}^{1\ldots 4} A_{s_1 s_2} dx_{s_1} dx_{s_2}$$

to be the integral of an exact differential.

Some of our readers may be interested in the general results for any number of dimensions. The proofs go along exactly as those given; there are two cases, according as p is even or odd (I_p is over the closed variety of p dimensions bounding the variety of $p+1$ dimensions over which I_{p+1} is extended).

A. p odd.

$$A_{s_1\cdots s_{p+1}}=\frac{\partial A_{s_1\cdots s_p}}{\partial x_{s_{p+1}}}-\frac{\partial A_{s_2\cdots s_{p+1}}}{\partial x_{s_1}}+\cdots-\frac{\partial A_{s_{p+1}s_1\cdots s_{p-1}}}{\partial x_{s_p}}.$$

B. p even.

$$A_{s_1\cdots s_{p+1}}=\frac{\partial A_{s_1\cdots s_p}}{\partial x_{s_{p+1}}}+\cdots+\frac{\partial A_{s_{p+1}\cdots s_{p-1}}}{\partial x_{s_p}}.$$

In each case

$$I_p\equiv\int\sum' A_{s_1\cdots s_p} dx_{s_1}\cdots dx_{s_p}$$

is equal to

$$I_{p+1}\equiv\int\sum' A_{s_1\ldots s_{p+1}} dx_{s_1}\cdots dx_{s_{p+1}}.$$

§ 2. APPLICATION TO MAXWELL'S EQUATIONS.

Consider that half of Maxwell's equations which in the usual notation and units are written

$$\operatorname{curl} \overline{E} + \frac{1}{c}\frac{d\overline{B}}{dt} = 0; \qquad \operatorname{div} \overline{B} = 0$$

and for the sake symmetry write (x_1, x_2, x_3, x_4) for (x, y, z, t). Then if we write

$$A_{12} = -A_{21} = B_z = B_3; \qquad A_{34} = -A_{43} = cE_z = cE_3$$
$$A_{23} = -A_{32} = B_x = B_1; \qquad A_{14} = -A_{41} = cE_x = cE_1$$
$$A_{31} = -A_{12} = B_y = B_2; \qquad A_{24} = -A_{42} = cE_y = cE_2$$

the first component

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{dB_x}{dt} = 0$$

of the vector equation may be written

$$\frac{\partial A_{34}}{\partial x_2} + \frac{\partial A_{42}}{\partial x_3} + \frac{\partial A_{23}}{\partial x_4} = 0$$

or in our previous notation $A_{234} = 0$. The other equations express the vanishing of A_{314}, A_{124}, A_{123} and so this group of Maxwell's equations simply says that

$$I_2 = \int \sum_{s_1 s_2}^{1\ldots4} A_{s_1 s_2} dx_{s_1} dx_{s_2} \equiv \int (B_x dydz + B_y dzdx + B_z dxdy) + c(E_x dxdt + E_y dydt + E_z dzdt)$$

is the integral of an exact differential—its value extended over any closed surface is zero. Just as in the static problem (which is a special case when the surface lies in the three-dimensional variety $x_4 = t = \text{const.}$) the surface integral $\int (B_x dydz + B_y dzdx + B_z dxdy)$ is zero when extended over any closed surface.

The remaining half of Maxwell's equations may be written

$$\operatorname{curl} \overline{H} - \frac{1}{c}\frac{d\overline{D}}{dt} = \bar{j}; \qquad \operatorname{div} \overline{D} = \rho,$$

where $\overline{D}$, the displacement vector, coincides with the electric force vector $\overline{E}$ in empty space, $\bar{j}$ is the current density, ρ the volume density of charge and $\overline{H}$ is the magnetic force vector. With the same notation as before for the integrals write

$$A_{12} = -A_{21} = -D_z; \qquad A_{14} = -A_{41} = c\overline{H}_x;$$
$$A_{23} = -A_{32} = -D_x; \qquad A_{24} = -A_{42} = c\overline{H}_y;$$
$$A_{31} = -A_{13} = -D_y; \qquad A_{34} = -A_{43} = c\overline{H}_z;$$

and the first component—

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{1}{c}\frac{dD_x}{dt} = j_x$$

of the vector equation may be written

$$\frac{\partial A_{34}}{\partial x_2} + \frac{\partial A_{42}}{\partial x_3} + \frac{\partial A_{23}}{\partial x_4} = cj_x$$

or

$$A_{234} = cj_x.$$

The other equations may be similarly written $A_{314} = cj_y$; $A_{124} = cj_z$; $A_{123} = -\rho$ and so this group of Maxwell's equations expresses that

$$\int c(H_x dxdt + H_y dydt + H_z dzdt) - (D_x dydz + D_y dzdx + D_z dxdy)$$
$$= \int c(j_x dydzdt + j_y dzdxdt + j_z dxdydt) - \rho dxdydz,$$

the surface integral being extended over the boundary of the *arbitrary* variety of three dimensions over which the triple integral is extended. In the static problem we have Gauss' Theorem which is usually expressed in the form $\int D_n dS = \int \rho dxdydz$; if we are careful to leave the metrical properties of the space undefined the symbol $\int D_n dS$ is simply an abbreviation for $\int (D_x dydz + D_y dzdx + D_z dxdy)$. Incidentally for any *closed* variety of three dimensions we have

$$\int c(j_x dydzdt + j_y dzdxdt + j_z dxdydt) - \rho dxdydz = \text{zero}.$$

For example the variety may be considered as traced out by a moving surface which may change its form continuously as it moves, and the vanishing of this integral simply says that the rate of accumulation of charge inside the surface is equal to the rate of flux of current through the surface. As a matter of fact from the equations

$$cj_x = A_{234} \equiv \frac{\partial A_{23}}{\partial x_4} + \frac{\partial A_{34}}{\partial x_2} + \frac{\partial A_{42}}{\partial x_3}, \qquad \text{etc.},$$

we obtain at once $c \operatorname{div} \bar{j} = \partial\rho/\partial t$ which is the analytical statement of what has just been said.

The Vector Potential.

The fact that $\int B_x dydz + B_y dzdx + B_z dxdy + c(E_x dxdt + E_y dydt + E_z dzdt)$ is the integral of an exact differential leads to the conclusion that its value, when extended over any open surface, depends merely on the boundary and it should, therefore, be equivalent to a certain line integral extended over that boundary. In fact let us write the surface integral equal to $\int (c\phi dt - (A_x dx + A_y dy + A_z dz))$—the minus sign being introduced merely for the purpose of maintaining a well-established

notation. Here $A_1 = -A_x$; $A_2 = -A_y$; $A_3 = -A_z$; $A_4 = c\phi$ and so, using Case I of Stokes' Theorem

$$B_x = A_{23} \equiv \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

There are two similar equations for B_y, B_z and all three may be grouped in the vector equation $\overline{B} = \text{curl } \overline{A}$. Further

$$cE_x = A_{14} \equiv \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} = -\frac{\partial A_x}{\partial t} - c\frac{\partial \phi}{\partial x};$$

there are two similar equations for E_y, E_z and we again group the three equations in the vector equation

$$\overline{E} = -\text{grad } \phi - \frac{1}{c}\frac{\partial \overline{A}}{\partial t}.$$

We have then six partial differential equations for the four unknown quantities A_x, A_y, A_z, ϕ. They are *consistent* owing to the fact that

$$\int (B_x dydz + B_y dzdx + B_z dxdy + c(E_x dxdt + E_y dydt + E_z dzdt))$$

is the integral of an exact differential. To make this clear we shall actually construct an integral as follows. Let us try for an integral where $A_z \equiv 0$. The equations

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}; \qquad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}; \qquad E_x = -\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}$$

yield

$$A_z = -\int_{x_0}^{x} B_y(xyzt)dx + A_z'(y, z, t),$$

$$A_y = \int_{x_0}^{x} B_z(xyzt)dx + A_y'(y, z, t),$$

$$\phi = -\int_{x_0}^{x} E_x(xyzt)dx + \Phi(y, z, t),$$

where x_0 is a constant of integration and A_z', A_y', Φ are, so far, arbitrary functions of y, z, t. On substituting in the three remaining equations and availing ourselves of the conditions for an exact differential—Maxwell's equations in fact—we obtain three equations of the same form as the six we started out with but involving merely *three* variables y, z, t instead of four x, y, z, t. For example from

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

we have

$$B_x(x, y, z, t) = \frac{\partial A_z'}{\partial y} - \frac{\partial A_y'}{\partial z} - \int_{x_0}^{x} \left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) dx$$

$$= \frac{\partial A_z'}{\partial y} - \frac{\partial A_y'}{\partial z} + \int_{x_0}^{x} \frac{\partial B_x}{\partial x} dx - \text{since div } \overline{B} = 0,$$

whence

$$B_x(x_0yzt) = \frac{\partial A_z'}{\partial y} - \frac{\partial A_y'}{\partial z}.$$

The remaining equations are

$$E_y(x_0yzt) = -\frac{\partial \Phi}{\partial y} - \frac{1}{c}\frac{\partial A_y'}{\partial t}; \qquad E_z(x_0yzt) = -\frac{\partial \Phi}{\partial z} - \frac{1}{c}\frac{\partial A_z'}{\partial t}.$$

As before we seek a solution of this simpler set such that $A_y' \equiv 0$ say—and so on. The solution obtained in this way is

$$A_x = 0; \qquad A_y = \int_{x_0}^{x} B_z(xyzt)dx;$$

$$A_z = -\int_{x_0}^{x} B_y(xyzt)dx + \int_{y_0}^{y} B_x(x_0yzt)dy;$$

$$\phi = -\int_{x_0}^{x} E_x(xyzt)dx - \int_{y_0}^{y} E_y(x_0yzt)dy - \int_{z_0}^{z} E_z(x_0y_0zt)dz,$$

where x_0, y_0, z_0 are constants of integration. The three quantities (A_x, A_y, A_z) form Maxwell's vector potential whilst ϕ is his scalar potential; from our point of view they are intimately bound together to form what is called the *electromagnetic potential*. It may be remarked that the explicit formula obtained is of little interest to the physicist; the electromagnetic potential is useful to him mainly when it can be calculated independently thus enabling him to derive the values of the electric and induction vectors. However it is necessary to show that there actually are functions A_x, A_y, A_z, ϕ with the characteristic property of potential functions—*i.e.*, such that the required vectors can be obtained from them by differentiation. It may be of interest to say a few words more about the line integral $\int c\phi dt - (A_x dx + A_y dy + A_z dz)$; linear differential forms—$\sum_{s_1}^{1\cdots 4} A_{s_1} dx_{s_1}$—have important applications in many branches of theoretical physics, analytical dynamics and hydrodynamics for instance; they have been extensively studied by the mathematicians under the name of Pfaffians. Suppose we are interested in the value of $\int \sum_{s_1}^{1\cdots 4} A_{s_1} dx_{s_1}$ over an open curve and ask are there any extremal curves?; in other words given two points does there exist a joining curve such that the integral over it is a maximum or minimum. By the application of the first principles of the calculus of variations

we find as necessary conditions

$$\sum_{s_2}^{1\cdots 4} A_{s_1 s_2} dx_{s_2} = 0, \qquad s_1 = 1, 2, 3, 4,$$

where, as before,

$$A_{s_1 s_2} = \frac{\partial A_{s_1}}{\partial x_{s_2}} - \frac{\partial A_{s_2}}{\partial x_{s_1}}.$$

We have here four homogeneous linear equations for $dx_1 \cdots dx_4$ and they are inconsistent unless the determinant of the coefficients vanishes. This determinant is skew symmetric and is, accordingly a perfect square, it being $(A_{12}A_{34} + A_{13}A_{42} + A_{14}A_{23})^2$. In the problem of the electromagnetic potential line integral, then, the condition for extremals is the vanishing of $(\bar{E} \cdot \bar{B})$. Fields of this type are quite special yet are of frequent occurrence especially in radiation problems. Now it is possible by a change of variables from $x_1 \cdots x_4$ to $(y_1 \cdots y_4)$, say, to reduce the linear differential form to a simpler one which is called the *standard* or *normal* form. There are two cases according to whether extremals do or do not exist:

(*a*) *Extremals exist.* $(\bar{E} \cdot \bar{B}) = 0$. Electromagnetic field is special. Standard form is $\sum_1^p L_s dy_s + dU$, p = zero or 2. The L's and U are functions of $y_1 \cdots y_4$ and therefore indirectly of the original space time variables x, y, z, t.

(*b*) *Non-existence of Extremals.* Standard form is $L_1 dy_1 + L_2 dy_2$.[1]

§ 3. Connection with the Absolute Differential Calculus.

The statement of Maxwell's equations in the form of relations between integrals lends itself at once to the methods of the absolute calculus, and it may not be out of place to mention here that Bateman, in his paper already quoted, had in this way obtained the general mode of transformation of the electric and magnetic force vectors prior to the publication of Einstein's researches on the *General Relativity Principle*. The differentials dx_s, $s = 1 \cdots 4$, form a contravariant tensor of rank 1; the products $dx_r dx_s$ form, from their definition a skew-symmetric contravariant tensor of rank 2; (it is the product of the difference of the two tensors

$$\frac{\partial x_r}{\partial u_1} \cdot \frac{\partial x_s}{\partial u_2} = \phi^{rs}$$

[1] It is impossible to give without unduly enlarging this paper proofs of the statements made with reference to linear differential forms. The reader is referred to a delightfully simple yet thorough exposition of the underlying mathematical theory by E. Goursat, Bull. de la Soc. Math. de France, t. 44 (1916), "Sur quelques remarques relatives au problème de Pfaff."

and

$$\frac{\partial x_r}{\partial u_2} \cdot \frac{\partial x_s}{\partial u_1} = \phi'^{rs}$$

by du_1, du_2, and this latter product of the differentials of the independent variables is invariant). Similarly $dx_r dx_s dx_t \equiv \phi^{rst}$ is a skew-symmetric contravariant tensor of rank 3. Consider then the integral

$$\int c(j_x dydzdt + j_y dzdxdt + j_z dxdydt) - \rho dxdydz$$

and let us make the relativistic demand of the invariance of form of physical equations, *i.e.*, when we change our coördinates from (x, y, z, t) to (x', y', z', t') what we shall have for current density in the new system if the integral is to keep its form is

$$j'_x = j_x \frac{\partial(yzt)}{\partial(y'z't')} + \text{etc.} - \frac{\rho}{c} \frac{\partial(xyz)}{\partial(y'z't')},$$

since

$$dydzdt = \frac{\partial(y, z, t)}{\partial(u_1, u_2, u_3)} du_1 du_2 du_3 = \frac{\partial(y, z, t)}{\partial(x', y', z')} dx'dy'dz' + \frac{\partial(y, z, t)}{\partial(y', z', t')} dy'dz'dt' + \text{etc.}$$

This gives the four quantities j_x, j_y, j_z, $(-\rho/c)$ as the four distinct functions of a covariant, skew-symmetric tensor of rank 3. It is only in the special relativity theory where the transformations are restricted to linear orthogonal transformations that this is equivalent to saying that these 4 quantities form a four vector.

Again from the equivalent surface integral we have that cH_x, cH_y, cH_z, $-D_x$, $-D_y$, $-D_z$ are the six distinct functions of a skew-symmetric covariant tensor of rank 2.

From the surface integral which is zero over every closed surface we get $(B_x, B_y, B_z, cE_x, cE_y, cE_z)$ as the six distinct functions of a skew-symmetric covariant tensor of rank 2 and then the invariance of the line integral gives $(A_x, A_y, A_z, -c\phi)$ as the four components of a covariant tensor of rank 1. That Maxwell's equations are tensor equations follows at once. For from any covariant tensor ϕ_{rs}, of rank two we get by differentiation a covariant tensor of rank 3

$$\phi_{rs,t} \equiv \frac{\partial \phi_{rs}}{\partial x_t} - \sum_g \left\{ \begin{matrix} rt \\ g \end{matrix} \right\} \phi_{gs} - \sum_g \left\{ \begin{matrix} st \\ g \end{matrix} \right\} \phi_{rg},$$

where we have used the ordinary notation for the Christoffel three index symbols of the second kind. By cyclic permutation of the suffixes, we get two other covariant tensors of rank 3

$$\phi'_{rst} \equiv \phi_{str}, \qquad \phi''_{rst} = \phi_{trs},$$

and adding up these three tensors we get a new covariant tensor of rank 3. In particular if ϕ_{rs} is skew symmetric this new tensor turns out to be

$$\frac{\partial \phi_{rs}}{\partial x_t} + \frac{\partial \phi_{st}}{\partial x_r} + \frac{\partial \phi_{tr}}{\partial x_s}.$$

Generalizing this, we see the tensor character of the conditions for an exact differential. Finally, one set of Maxwell's equations express the vanishing of the skew-symmetric covariant tensor of rank 3 formed in this way from the skew-symmetric covariant tensor of rank 2 $(\overline{B}, c\overline{E})$; and the other set, the equivalence of a skew-symmetric covariant tensor of rank 3 formed in this way from the skew-symmetric covariant tensor of rank 2 $(c\overline{H}, -D)$ to the current skew-symmetric covariant tensor of rank 3.

JOHNS HOPKINS UNIVERSITY,
July 1, 1920.

THE FORMATION OF NEGATIVE IONS IN AIR.

By Leonard B. Loeb.

Synopsis.

Mobility of Negative Ions in Air at Low Pressures.—The purpose of the experiments was to decide between the two theories of the formation of negative ions, proposed by Wellisch and by J. J. Thomson. The carriers were generated as photo-electrons by focusing ultra-violet light on one plate of a condenser at grazing incidence, thus eliminating stray light effects. Great care was used to secure pure air. The *curves* obtained agree in general with the results of previous investigators; they show anomalous mobilities below a critical pressure which was found to vary with the frequency of commutation of the E.M.F. No "free electrons" of the Wellisch type were observed. A repetition of the Wellisch experiments with photo-electrons showed that the "free electrons" were really a type of carrier which, started as electrons but attached themselves to molecules in the measuring field, as the Thomson theory demands. To *test the Thomson theory*, curves were computed assuming certain values for the constants. Theoretical and observed curves are similar in shape; in particular, the inflections near the low voltage end fall fairly closely together; It is also shown that the Wellisch *results* are in accord with this theory. The *chance of negative ion formation constant*, $1/n$, comes out about $1/2.5 \times 10^6$ *for air*. As $1/n$ is 0 for *nitrogen*, and as $1/n$ in *oxygen* is about 1/50,000, we conclude that it is to the oxygen molecule in air that electrons attach to form ions.

Introduction.

It has long been known that the velocity in unit electric field (*i.e.*, mobility) of normal negative ions in air ceased to be strictly inversely proportional to the pressure, for pressures below 10 cm. of air. In fact for pressures lower than this the mobility has been found to increase more rapidly than the pressure decreased. This behavior has been taken as powerful evidence for the existence of so-called "cluster" ions, but the work of Wellisch (1), Loeb (2), and Yen (3) has shown that the assumption of a fairly stable cluster ion must be given up. One must accordingly look elsewhere for an explanation of the foregoing phenomenon.

In most if not in all ionization processes in gases the first step consists in the liberation of an electron from a neutral atom or molecule. As all evidence points to the fact that the normal negative ion in a gas is a body of molecular dimensions, the electrons formed by the ionizing agent must first attach themselves to neutral gas molecules in order to form the type of carriers that are usually observed. It seems quite likely that in this process may lie the explanation of the abnormal mobilities of gas ions observed in air at low pressures.

In fact Sir J. J. Thomson (4) proposed a theory to account for these abnormal, or better "anomalous," mobilities, on the basis of a definite mode of formation of ions from electrons and neutral molecules. This theory assumes that out of n collisions of an electron with a given type of neutral molecule on the average only one will result in the attachment of the electron to the molecule: n being a constant characteristic of the type of gas molecules considered. Such an assumption might be interpreted as meaning that a certain region of the molecule must be struck by the electron under the proper conditions (*e.g.*, such as the velocity of the electron or the state of the molecule) before it can be incorporated in the molecule. It is obvious that if n be great enough, particularly at low pressures, the electron might cover considerable distances as an electron before attaching itself to a molecule to form an ion. Thus since the mobility of the electron is of the order of a hundred fold that of the ion it would be expected that below certain pressures the mobility of the negative ions would show abnormalities. On the basis of this theory J. J. Thomson derived the equations governing the behavior of carriers which change their nature while being measured as a function of the pressure. His computations are applicable to the Langevin method of mobility determination.

In two very able papers Wellisch (1) published the results of an investigation of the mobilities of positive and negative ions at low pressures in air as determined by the Franck (5) modification of the Rutherford alternating current method. Down to 8 cm. he observed the single class of carriers of slightly abnormal mobilities observed by the earlier workers (5). Below 8 cm. pressure his curves were of such a form that it seemed only possible to explain them on the basis of the existence of *two types of carriers*. These Wellisch interpreted as being carriers of mobilities approaching electronic magnitudes, which he termed "free electrons," and normal ions whose mobility was inversely proportional to the pressure down to about .15 mm. The interpretation of the process of ion formation given by Wellisch, as a result of his experiments (which yielded two types of carriers at pressures where other observers had found but a single type), differs radically from that offered by Thomson. Assuming that in air the carriers are of two distinct kinds, "free electrons" and ions, the former increasing in numbers relatively to the latter as the pressure decreases, Wellisch proposed the following theory. In order that an electron may attach to a molecule to form a negative ion he assumed that the electron must strike the molecule with a relative velocity, corresponding to a certain energy-value greater than E, which he styles the "potential energy of ion formation." If the

energy at impact is greater than E the electron and molecule will unite to form an ion. If the energy of the impact be less than E the electron can never attach to the molecule. It then becomes a "free electron," which it remains unless it can acquire the energy E in the electric field. Now this energy E he further assumes is generally acquired in the act of ionization. In receding from the positive remainder of the molecule from which it came the electron transfers some of its initial kinetic energy to potential energy of separation. Accordingly if before striking its first molecule it has traveled so far that its residual energy is less than E it will remain a "free electron." The number of electrons which have a chance of thus remaining free will obviously be the greater the lower the pressure, for the mean free path of an electron is inversely proportional to the pressure.[1]

One has therefore two widely different theories of ion formation. The theory of Thomson assumes that out of a great many collisions the electron may strike a molecule in such a manner that it can attach, an event which if one is given enough impacts must eventually take place *for all electrons*. The theory of Wellisch postulates that it is the first impact that determines whether an electron shall form an ion or whether it shall pursue its subsequent career as a *permanently* "free electron." These theories represent different interpretations of the results of two slightly different methods of experimentation, which find reconciliation in the light of the results which follow.

THE PROBLEM.

In studying this question it became evident to the writer that no decision as to the validity of the two theories could be made without a further experimental investigation. Accordingly in October, 1919, a series of experiments were undertaken in order to attempt to solve this problem. What was needed was a method of distinguishing sharply between electrons and ions in experiments of the type of the earlier investigators. It was thought possible so to modify the general method used by Kovarik (6) as to make it serve this purpose. Kovarik measured the mobilities of carriers produced by photo electrons generated at the surface of a plate by light incident normally upon it, for various pressures in air, using the Rutherford alternating current method. In again taking up experiments of this sort an attempt was made to fulfil the following conditions.

[1] It is to be remembered that the force between the parent atom and the electron escaping from it is inversely proportional to the square of the distance and not to a relatively high power of the distance as in cohesion.

1. To use a detecting system having a high sensibility in order to detect the presence of small numbers of free electrons.

2. To use carriers all starting from the same plane as electrons.

3. To use fields as uniform as possible.

4. To control the upper limit of the velocity of emission of the photo-electrons, by filters if necessary.

5. To eliminate the effect of "stray light" which Kovarik claims to have been bothered by.

6. To eliminate all *organic vapors* and *impurities*.

7. To use a constant source of ultra-violet light.

APPARATUS.

The apparatus finally evolved is indicated in Fig. 1. In an all metal case C are housed the two condenser plates P and E of 10 cm. diameter.

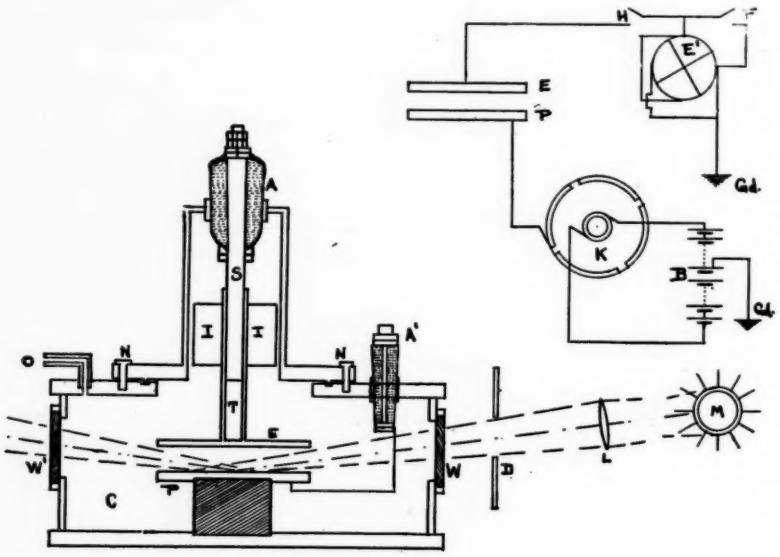


Fig. 1.

P was a plate of highly polished speculum metal which seemed to have quite suitable qualities as a constant source of photo-electrons. E was a plate of oxidized copper suspended by means of the threaded tube T so as to be approximately parallel to P. E could be screwed up or down on the threaded support S, by means of an electromagnet operated from outside the housing which acted on the iron mass I attached to T. S was insulated by an amber plug A ground into the top of the housing case C, and was attached to the electrometer system. The image of a quartz mercury arc M was focused upon P through a quartz window W, 4.5 cm.

in diameter, by means of the quartz lens L. From P most of the light was reflected out through a glass window W through which also the distance PE could be accurately measured by means of a cathetometer. The interior of C was oxidized so that the stray light falling on the walls of the housing emitted practically no electrons. The beam of light was still further cut down by the diaphragms D. Access into C was gained through the removal of the top of the case C, by unscrewing the screws N, which insured the tightness of the joint. This joint was made gas tight through the use of an unvulcanized rubber gasket, similar to those used by Professor Millikan in the attainment of high vacuua on his vacuum spectrometer.

In sealing in the windows of the chamber C as well as in securing gas tight joints at the amber insulating plug A^1, which carried the leads from the plate P to the source of the alternating current, great care was taken to make the possibility of the diffusion of any vapors from these points into the case negligible.

The electrometer E^1 was one of the Dolezalek type having with its system of switches and leads an electrostatic capacity of 260 E.S.U. and a sensibility of 3,700 mm. per volt on a scale 2 m. distant. The alternating potential came from a bank of dry cells B the center of which was grounded at Gd, the other poles of which were connected to the two brushes of a commutator K. Two commutators were used, one having twenty segments, and the other one having two segments. They were driven by a motor through sets of gears having the ratios 5 : 1, 2 : 1, and 1 : 1. The speed of the motor was 2,100 R.P.M. so that by changing gears and commutators a fairly complete control of frequency of alternation lying between 15 and 750 cycles per second was obtained. The speed of the commutator was measured directly by means of a revolution-counter and stopwatch. The speed was kept constant by regulating the field current of the motor. Small variations in the speed of the motor were registered by variations in the E.M.F. given by a small dynamo driven by the shaft of the driving motor. This indicator was so sensitive that any desired degree of constancy could be obtained.

The air used in the measurements was purified in the following manner. It passed through a tube of copper oxide heated to dull redness to destroy any organic vapors present. The air then passed through long tubes of NaOH, $CaCl_2$, and P_2O_5, finally passing through a trap cooled to from $-125°$ to $-90°$ C. by frozen alcohol. Before any series of measurements on air was made, the apparatus was exhausted to about 2 cm. and filled to atmospheric pressure with the purified air at least three times, generally more.

THE MEASUREMENTS.

Measurements were conducted for a given pressure and a given frequency of commutation by determining the variation of the current to the electrometer plate as a function of the value of the alternating potential applied to P. This was accomplished by disconnecting the lead to the electrometer at switch H, leaving the latter grounded through F and allowing the alternating potential to act on plate P for a given time (from 5 to 30 seconds). The electrometer ground F was then disconnected, the plate E connected to the electrometer, and the deflection noted. Since the capacity of the leads to the electrometer was appreciable compared with the capacity of the condenser formed by the plates P and E, a correction of about 7.5 per cent. had to be subtracted from the value of the P.D. given by the commutator, in order to give the true value of the P.D. acting across P and E.[1] In general the potentials taken from the cells were so arranged that the value of the positive or retarding, potential applied to the plate P by the commutator, was always from 2 to 20 per cent. higher than the corresponding negative, or accelerating potential. The purpose of this was to prevent any possibility of carriers *working* their way across from P to E by stages covering several cycles. Control experiments showed that the magnitude of this difference in the two sides of the alternating P.D. did not play any rôle as long as the retarding potential was a trifle greater than the accelerating potential. The values of the deflections for various potentials were generally taken starting at the higher values and going down to the lowest, then working back again over the curve taking values of the potentials at intermediate points between those first chosen. The readings were always quite consistent, unless by some accident the value of the potential across the mercury arc had changed during the measurement. For a series of readings on a given sample of air the curve at atmospheric pressure would be taken first, and then the values for the lower pressures were obtained, the pressure being reduced step by step, by means of a water aspirator. The range of pressures over which satisfactory work could be done in air, lay between 760 mm. and 30 mm. Below this pressure it was impossible to go because the frequencies attainable by the commutator were not high enough for the type of curves obtained. The frequencies used varied from 15 cycles per second to 750 cycles while the distances between P and E varied

[1] The total fall of potential given by the commutator was distributed across the condenser formed by P and E, and the condenser formed by the leads and the ground, these two representing condensers coupled in series. The true field acting on the carriers was then less than that produced by the commutator. Measurements made by the method of mixtures gave this correction as 7.5 per cent.

from 1 cm. to 1.7 cm. At some point in each series of determinations, for a given sample of air a "control" set of readings was taken in which the deflection of the electrometer was measured when different steady negative potentials were applied to the plate P, for a given time. This gave the saturation photo-electric current from the plate P under the influence of the arc at the time of experiment. Knowledge of this current served as a means of comparing data taken on successive days. It was also of use in applying the theory of Thomson.

When the electrometer deflections for a given series of accelerating potentials were plotted against the value of the accelerating potential, curves of the type shown in Fig. 2, curves I., II., III., IV., and V., were

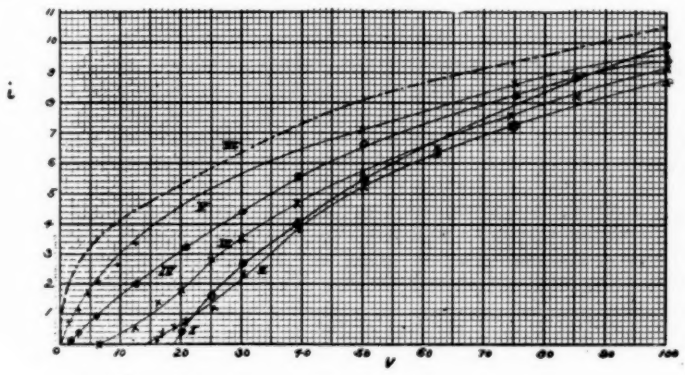


Fig. 2.

June 2, 1920. $d = 1.05$. Electrometer deflections in cm. scale reading plotted against the alternating P.D. on P.

I.	$p = 747.0$	$N = 35.7$	Mobility	$K' = 2.27$
II.	$p = 98.0$	$N = 359.0$	"	$K' = 2.97$
III.	$p = 75.5$	$N = 705.0$	"	$K' = 7.92$
IV.	$p = 53.5$	$N = 705.0$	"	$K' = 26.0$
V.	$p = 38.5$	$N = 705.0$	"	$K' = 39.4$
		Observed x...x...		
VI.		Saturation −.−.−.		

obtained. These correspond to different pressures and to different speeds of commutation. The curve numbered VI. in the figure is the control or saturation curve taken with a series of steady negative potentials applied to plate P. The curves in general have the characteristic form of mobility curves. They are nearly parallel to the saturation curves at the higher potentials, and then drop sharply towards the potential axis tending to cut it at some fairly well defined point. The voltage value of this point one may designate as V_0. In the curves obtained at the higher pressures this point is quite sharply defined.

If the value of V_0, obtained from a given curve and corrected for the

potential drop across the electrometer leads by being reduced by 7.5 per cent. of its value, be inserted as V_0 in the equation $U = Nd^2/V_0$ (where N is the frequency of commutation in half cycles, and d is the distance between the plates), the average value of U the mobility of the carriers between the plates is obtained. If the U thus obtained for a given pressure, be multiplied by the factor $p/760$ (where p is the pressure in mm.), the value of K the *mobility constant* for normal ions in air should be obtained. The values of K for the curves depicted in Fig. 2 may be computed from the data given in the legend. They are 2.27, 2.97, 7.92, 26 and 39.4. cm. per second respectively for curves I., II., III., IV. and V., Fig. 2.

RESULTS.

The results of twenty-five different series of determinations, taken under all imaginable conditions in air, may be summed up as follows:

1. No evidence of the existence of the *two* types of carriers described by Wellisch could be found in these results, although they were carefully looked for; nor was there any evidence of the "free electrons" of Wellisch. Such carriers would have been indicated by complex curves of the types shown in Figs. 14 and 15, taken under different circumstances.

2. The curves were consistently of the types shown in Fig. 2 having intercepts with the voltage axis V_0 which above 100 mm. pressure yielded approximately normal values of K. The values of V_0 obtained at pressures below this gave anomalous values for K which were in general accord with the values found by Kovarik.

3. The value of K for carriers below 100 mm. pressure was found to be a conspicuous function of the frequency of commutation. This is again in agreement with the experimental results of Kovarik.

4. The anomalous values of K for air were not always the same under similar conditions but varied somewhat with the purification which the air had undergone.

5. As may be seen in the figures, the apparently sharp intercepts of the curves with the potential axis begin to vanish below about 100 mm. and the feet of the curves tend to approach the axis more and more asymptotically the lower the pressure (see especially II. and III.). The appearance of such feet in the curves was also noted by Kovarik who ascribed them to "stray light" effects. A careful study of these portions of the curves which appear in all curves when the pressures become low enough, under all conditions of illumination and plate distance, is sufficient to show that *they are intimately connected with the phenomenon of the anomalous mobilities and are not due to stray light.*

It may be concluded from the above summary that these results verify

the experimental results of the early observers. They however go a step further than those results indefinitely connecting the asymptotic portions of the curves near the potential axis with the phenomenon of the anomalous mobility. The work was carried out in such a manner that it permits an attempt at the quantitative application of the results to the theory of J. J. Thomson.

APPLICATION OF RESULTS TO THOMSON THEORY.

For carriers liberated at one plate P of a parallel plate condenser, and accelerated for a time T to the other plate E distant d, by a potential difference of value V only those carriers will reach E which have for a velocity in unit field a value U given by

$$U \geqq d^2/VT \quad (1).$$

Now the theory of J. J. Thomson assumes that an electron starting from P and moving towards E in the electric field may on its career in one out of n impacts collide with a gas molecule in such a manner as to attach itself to the molecule, and thus continue its way to E as an ion. A carrier which reached E after undergoing such a change must have had an average, "hybrid" mobility U which is determined by the condition laid down above. The value of U for such a carrier would then lie between that of an electron and that of an ion. In other words it would have a "fictitious" abnormal or an anomalous mobility such as is observed for carriers in air at pressures below 100 mm. Assume that such a carrier travels x cm. as an electron and $(d - x)$ cm. as an ion. If K' be the mobility of an electron and K be that of an ion the time T taken to travel d in unit field will be

$$T = x/K' + (d - x)/K.$$

Since however K' is of the order of 100 times as great as K, one may for all practical purposes neglect the first term and write

$$T = (d - x)/K.$$

Since $U = d/T$ one may write

$$U = \frac{Kd}{(d - x)},$$

which expresses the fictitious hybrid mobility, in terms of the mobility of the ion and the ratio of the whole distance to the distance traversed as an ion. In order to get a current from P to E with an alternating current whose time of alternation is T one must impose the condition (1), which gives:

$$U = K\frac{d}{(d - x)} \geqq d^2/VT.$$

This implies that in order to reach E at all the carrier must have covered a distance x as an electron, x being governed by the equation

$$d - KVT/d \leqq x. \tag{2}$$

This condition can only hold up to $(KVT)/d = d$, for negative values of x have no meaning.

On the basis of the assumptions quoted, and further assuming that *the velocity of drift of the electrons in the electric field is small compared to their velocity of kinetic agitation*, Thomson shows that out of I_0 electrons starting from P the number I which go a distance x without uniting to form ions, is given by the equation

$$I = I_0 e^{-Wx/(nK'\lambda V/d)} \tag{3}$$

(where W is the mean velocity of kinetic agitation of the electrons, d the distance between the plates, K' the mobility of the electron, λ the mean free path of the electron, and $1/n$ the chance that any collision of an electron with a molecule will result in the formation of an ion). Now the carriers that have traveled a distance x without combining to form an ion can only be detected when they contribute to the current from P to E, so that in order that the above equation (3) may apply to the carriers detected, one must include the condition imposed by (2), viz., that

$$x \geqq d - KVT/d,$$

where x can have only positive values. Putting this condition into (3) one obtains

$$I/I_0 = e^{\frac{-dW}{nK'V\lambda}(d-KVT/d)}. \tag{4}$$

Which gives the fraction of the initial electron current starting from P that reaches the plate E as a function of the experimental variables d, V and T. Furthermore since K is inversely proportional to the pressure, and assuming K' and λ to be so (*i.e.*, to be proportional to $760/p$, where p is the pressure in mm. at which the measurements are made), and finally writing $T = 1/N$ the equation (4) becomes

$$I/I_0 = e^{\frac{-W}{nK'\lambda}\left(\frac{d^2(p/760)^2}{V} - \frac{K(p/760)}{N}\right)}. \tag{5}$$

If one assumes that the electrons are in thermal equilibrium with the gas molecules he may evaluate W. Taking λ for the electrons as $4\sqrt{2}$ times that of the molecules, and assuming that K' for electrons is about 200 (1, 7) while K for ions is about 2, the equation becomes applicable to experimental verification and the determination of the constant n, namely it takes the form

$$I/I_0 = e^{\frac{-9.9\times10^8}{n}\left(\frac{d^2(7/760)^2}{V} - \frac{(2p/760)}{N}\right)}. \tag{6}$$

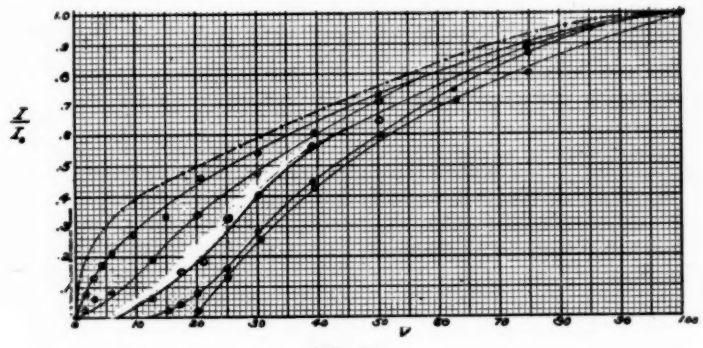


Fig. 3.

June 2, 1920. $d = 1.05$. Curves of Fig. 2 reduced to fraction of saturation.

I. $p = 747.0$ $N = 35.7$
II. $p = 98.0$ $N = 359.0$
III. $p = 75.5$ $N = 705.0$
IV. $p = 53.5$ $N = 705.0$
V. $p = 38.5$ $N = 705.0$
Observed x...x...
VI. Saturation −.−.−.

On the basis of the experimental curves obtained one may solve for n from any curve by taking the values of I/I_0,[1] d, p, V and N under which the curve was determined, and substituting them in the equation above.

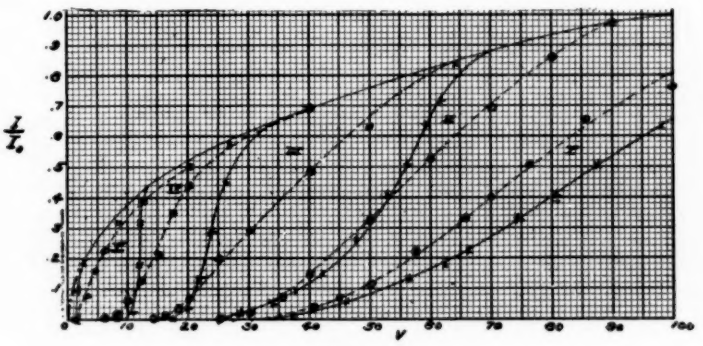


Fig. 4.

April 23, 1920. $d = 1.66$, $K = 2.1$, $p = 95$, $n = 2.5 \times 10^5$.

I. $N = 710$ III. $N = 362$
II. $N = 147$ IV. $N = 72$
V. $N = 14.7$
Observed o...o...
Computed -x——x-
Saturation −.−.−.

[1] I/I_0 was taken as the ratio of the electrometer deflection at a given voltage V taken with alternating potentials to the deflection obtained by letting the same steady voltage act for half the time. This gives the ratio of the true I/I_0 corrected for the decrease in the saturation current at the lower potentials.

In doing this it was found advisable to take points on the curves lying near the feet (*i.e.*, at values of V near those giving the abnormal mobilities for the curves). The reason for this will become evident later on. Solu-

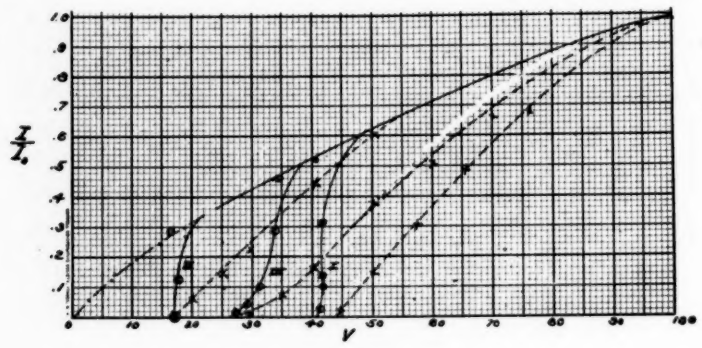


Fig. 5.

April 27, 1920. $d = 1.66$, $K = 2.45$, $n = 2.5 \times 10^5$.

I. $p = 738$ $N = 37.2$
II. $p = 738$ $N = 15.2$
III. $p = 148$ $N = 150.0$
Observed x...x...
Computed –o——o–
Saturation –.–.–.–.

tions of this equation for some fifteen or more different curves were made at one time or another. The curves so studied covered a range of pressures from 130 mm. down to 20 mm. in pressure, and frequencies ranging

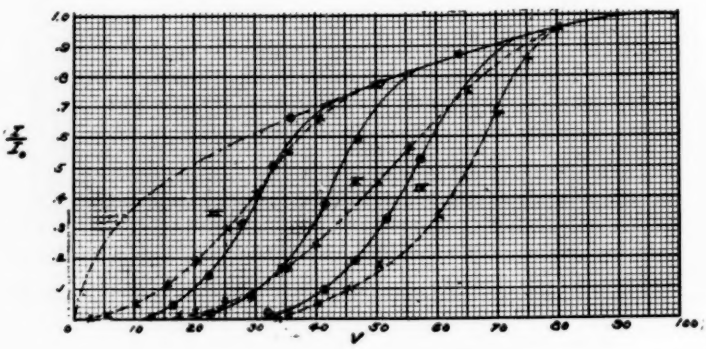


Fig. 6.

April 27, 1920. $d = 1.66$, $K = 2.45$, $n = 2.50 \times 10^5$.

IV. $p = 105$ $N = 372$
V. $p = 84$ $N = 378$
VI. $p = 59.2$ $N = 378$
Observed x....x...
Computed –o——o–
Saturation –.–.–.–

from 150 cycles to 750 cycles. The determinations also utilized three different values of d ranging from 1.05 cm. to 1.66 cm. and were taken on at least five different samples of air purified under somewhat different

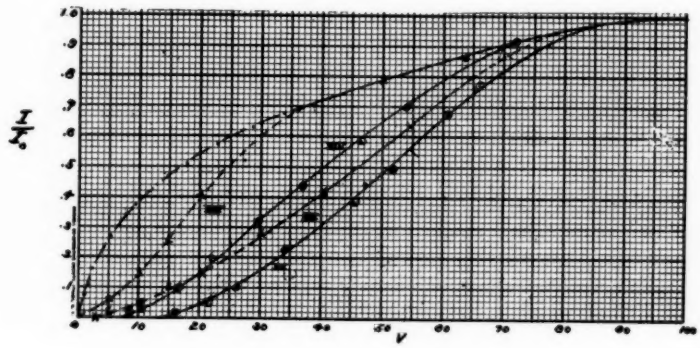


Fig. 7.

April 27, 1920. $d = 1.66$, $K = 2.45$, $n = 2.5 \times 10^5$.

VII. $p = 59.2$ $N = 742$

VIII. $p = 41.5$ $N = 742$

Observed x...x...

Computed -o——o-

Saturation —. —. —.

conditions. The values of n for air thus obtained ranged from 9×10^5 to 1.5×10^5, the greater portion of the values lying around 2.5×10^5. The reason for this variation will be considered in connection with a study of the curves. Measurements were made in pure oxygen, and in

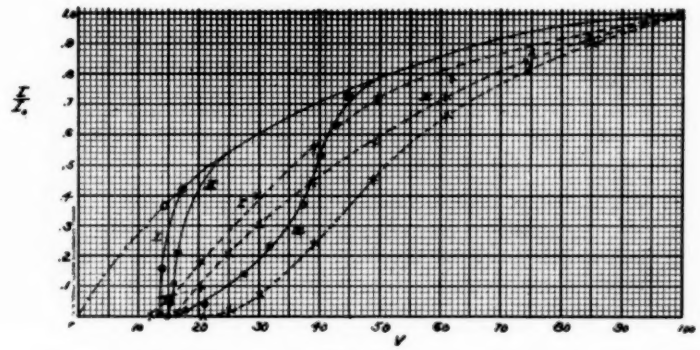


Fig. 8.

June 24, 1920. $d = 1.50$, $K = 2.55$, $n = 4 \times 10^5$.

I. $p = 752$ $N = 14.9$

II. $p = 198.5$ $N = 73.0$

III. $p = 99.5$ $N = 373.0$

Observed x...x...

Computed -o——o-

Saturation —. —. —. —.

nitrogen containing small percentages of oxygen, and will form the basis of a later paper. They covered pressures ranging from 760 mm. pressure to 15 mm. as well as frequencies from 37 cycles to 750 cycles and yielded

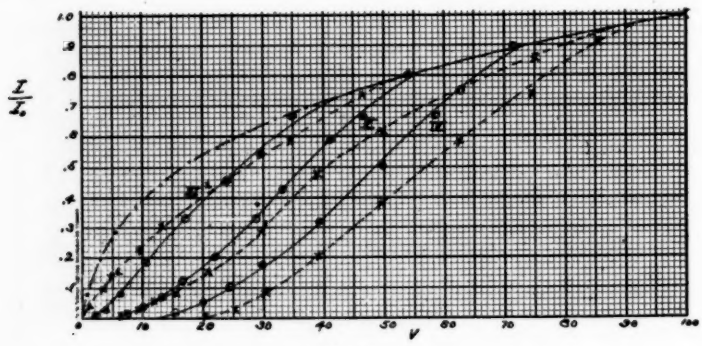


Fig. 9.

June 24, 1920, $d = 1.50$, $K = 2.55$; $n = 4 \times 10^5$.

IV. $p = 80.5$ $N = 727$
V. $p = 59.5$ $N = 740$
VII. $p = 32.5$ $N = 735$
Observed x...x...
Computed –o——o–
Saturation –. –. –. –.

values of n for air, computed back to air on the basis of the oxygen content, which agreed well with those already cited.

Using values of $n = 2.5 \times 10^5$ and 4×10^5 and using values of K, the ionic mobility, as calculated from the curve taken at 760 mm. (which

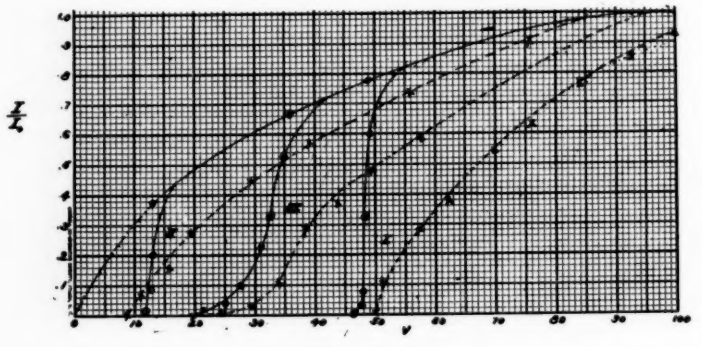


Fig. 10.

April 6, 1920. $d = 1.66$, $K = 2.16$, $n = 4 \times 10^5$.

I. $p = 744$ $N = 37.2$
II. $p = 207$ $N = 37.7$
III. $p = 134$ $N = 151.0$
Observed x...x...
Computed –o——o–
Saturation –. –. –. –.

ranged from 2.55 to 2.16 cm./sec.) the values of I/I_0 were computed as a function of V from the equation (6) for pressures and values of N under which some of the experimental determinations were made. The values of I/I_0 plotted as ordinates against the voltage V as abscissæ yielded a series of curves. These computed curves when corrected for the value of the saturation photo-electric current, and for the value of the reduction of potential between the plates due to the induction effect, are reproduced in Figs. 4 to 12 as full curves with circles. In the same figures are

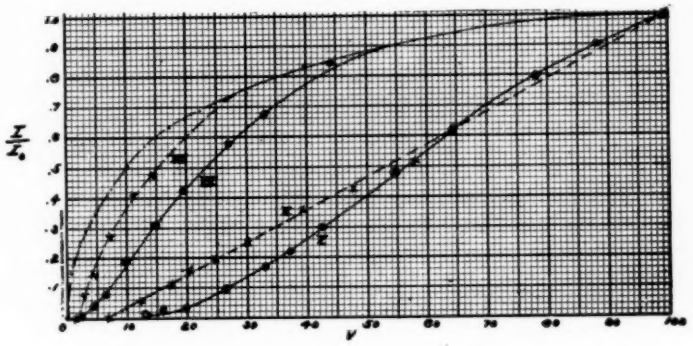


Fig. 11.

April 6, 1920. $d = 1.66$, $K = 2.16$, $n = 4 \times 10^5$.
V. $p = 76.5$ $N = 726$
VII. $p = 33.0$ $N = 742$
Observed x...x...
Computed –o——o–
Saturation –.–.–.–.

given, as dotted curves with crosses, the corresponding values of I/I_0 plotted against the voltage V as actually determined experimentally.[1] The curves represented by the dash dot lines, with no points marked, are in each case the actual saturation photo-electric current curves taken at the time of the experiment.

[1] The values of I/I_0 for the experimental curves were determined from the experimental curves by reducing the electrometer deflections actually measured to fractions of the apparent "saturation" value of the current. For measurements where the curves struck the axis below 40 volts this saturation value of the current was generally taken as the deflection at 100 volts accelerating potential. In some cases however it was taken from the half value of the "control" deflection at 100 volts. The two are closely the same for at 100 volts under most conditions of measurement the number of the carriers reaching the plate E in a given time of exposure to the alternating potential is practically equal to the number that reach E at the same voltage when the direct accelerating potential is on for half the time. The reason for reducing the readings with the direct potential to half is that with the alternating accelerating potential the field is on for but half the time of exposure. A typical transformation of the curves actually obtained may be seen in the case of the curves of Fig. 3 which are the curves of Fig. 2 as experimentally obtained reduced in the manner above indicated. In any case the changes produced near the feet of the curves by such a transformation are small and it is these parts of the curves which are of greatest importance.

The curves shown in Fig. 4 are the observed and the computed curves obtained at a pressure of 95 mm. with different frequencies. Figs. 5, 6 and 7, give the observed and computed curves taken on another sample of air with varying pressures and frequencies. The value of n used in the computation of both these sets of curves was 2.5×10^5. In the Figs. 8 and 9, as well as in the Figs. 10, 11 and 12, are given the observed

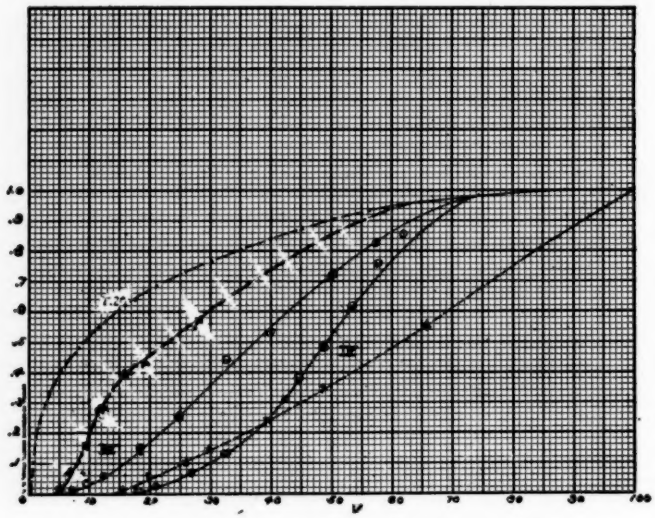


Fig. 12.

April 6, 1920. $d = 1.66$, $K = 2.16$, $n = 4 \times 10^5$.
IV. $p = 97$ $N = 376$
VI. $p = 55$ $N = 742$
Observed x. . . x. . .
Computed –o——o–
Saturation –. –. –.

and computed curves for two sets of determinations in air at varying frequencies and pressures, whose computed curves were obtained assuming a value of $n = 4 \times 10^5$. In all cases the legend gives the conditions under which the determinations were made, and the points illustrated are the actual points obtained. A number of other sets of determinations have been computed and compared with the experimental results with the same degree of success.

In examining the curves two points stand out prominently. First barring the upper portions of the curves taken at or near atmospheric pressure (where for reasons to be given later the curves should not agree) there is fair agreement between the general forms of the observed and computed curves: That is to say there is sufficient similarity in the form of the curves and in their variations with r and N to justify the

assumption that they correspond to an equation of the type used. Secondly it can be observed that the values of the voltage at which the ratio I/I_0 reaches some definite low value (*e.g.*, $I/I_0 = .02$ near the limit of measurement of the electrometer), are quite closely the same for the observed and the corresponding computed curves. *Since it is these portions of the experimental curves* (which one may term the feet of the curves), *which yield the values of V_0 corresponding to abnormal mobilities, it is obvious that the curves deduced on the basis of the results of the theory of J. J. Thomson yield values of V_0 quite well in accord with those actually observed.* The theory of Thomson then is in a certain measure well able to predict the anomalous mobilities found for the negative ions for low pressures, which have been the cause of so much discussion.

The reason for the failure of the curves to fit at their upper portions is as follows: At first sight one would expect the curves which are obtained at atmospheric pressure (where the carriers are for the most part normal ions from the beginning), to follow the saturation current curve with decreasing voltage until a value of V is reached, at which in the time of an alternation the carriers can just cross from the plate P to E. At this voltage, V_0, the current should abruptly fall to 0. This condition is implied in the application of the Thomson theory for in the development of that theory it is assumed for simplicity that as soon as V is such that $V = d^2/uT$ *the ions of mobility u all get across.* This is not the case in practice. Starting at a value of the alternating potential equal to V_0 in the negative phase on P, only those carriers will get across to E that were liberated in the first instants of that phase. For carriers liberated in the later portions of that phase will not have time enough to reach E under the existing field. On the reversal of the field such later ions will be all dragged back to the plate P and lost. As V is increased carriers that are liberated later, and later in the negative phase of the potential will succeed in crossing. So that the current to E will approach a sort of saturation value when V_0 has been considerably exceeded. The experimental curve would then not be expected to rise nearly vertically at V_0 as assumed, but would be expected to *begin* to rise at this value and then to gradually increase reaching a maximum at values of V a good deal higher than V_0. Such, as is well known, is actually the case.

In the case of the Franck modification of this method the curves are more like the theory, for only a part of the ions which are driven back to the plane of the gauze are absorbed by it while a plate, as distinguished from a gauze, will absorb them all.

Because of the presence of carriers which are electronic for a considerable portion of their path between the plates as the pressures are

reduced below 200 mm. the curves obtained on the basis of the Thomson theory with its simplifying assumption become themselves increasingly inclined to the vertical below this pressure. This has for a result that the differences in shape of the computed and the observed curves becomes much less marked at the lower pressures, as is readily seen.

There is however another point in which the curves do not conform. This is noticeable in Figs. 5, 6, and 7, where the constant n was taken as 2.5×10^5. While the feet of the curves at pressures above 59 mm. coincide quite well for the theoretical and the observed results, it is seen that the feet of the experimental curves below 59 mm. lie at voltages well *below* those for the computed curves. In other words the "anomalous" portions of the experimental curves actually seem to develop more rapidly with decreasing pressures than the theory would predict. This tendency is seen in the curves represented in Figs. 8 to 12, where the value of n chosen was 4×10^5. In this case the feet of the curves at 59 mm. coincide reasonably well, while the feet of the experimental curves taken for 80 and 100 mm. lie at *higher* voltages than the corresponding theoretical curves. In other words the *higher* value of n causes a closer agreement between the curves at the *lower* pressures, than at the higher ones. Accordingly n does not seem to be constant but appears to increase with decreasing pressure. This was the result obtained throughout in computing n. The curves at the lower pressures from 20 mm. to 60 mm. yielded values of n from 9 to 4×10^5, while the curves for pressures between 60 mm. and 130 mm. yielded values of n ranging from 4.0 to 1.5×10^5. The value of n was computed from the theory, and is subject to the condition that the assumptions of the theory are fulfilled. Now J. J. Thomson assumed that the velocity of the electron produced by the field between the plates was *negligible* compared to the velocity w which it possessed in virtue of its kinetic energy of agitation. Now w at 20° C. is about 1.2×10^7 cm./sec., while the velocity acquired between impacts in a field of 10 volts per cm., at 30 mm. pressure lies in the neighborhood of 7.1×10^6 cm./sec., *i.e.*, it is about half of w. If in addition it be considered possible that the velocity thus acquired in the field is not completely wiped out at each impact (*e.g.*, due to quasi-elastic impacts between electrons and nitrogen molecules) one sees how at considerably higher pressures the conditions assumed by the theory may not be met. Such elastic impacts were observed between electrons and hydrogen molecules by Franck (8) and Hertz, and Compton (9) states that they may also be possible in the case of impacts with oxygen molecules though to a much less degree than in hydrogen. In fact Wellisch (1) uses this concept as a possible explanation of some peculiar

results obtained in studying the mobilities of electrons. The writer in assuming the value of K' to be a constant of about 200 cm./sec. in value introduced any variation in K', due to the causes above considered, into n. One must then ascribe the deviations of these results from the theoretical ones not to variations in n but to the causes above outlined. In choosing a value for n for future reference it is best, in view of this increase of n with lower pressures, to choose the value of n which was determined at the lower range of pressures (*i.e.*, $n = 2.5 \times 10^5$), where less errors are introduced into the assumptions.

From the above curves one is justified in concluding that *these results furnish a strong qualitative verification* of the J. J. Thomson theory of ion formation.

CORRELATION OF THOMSON THEORY WITH RESULTS OF WELLISCH.

Having now obtained evidence for the correctness of the Thomson theory it remains to correlate these results and the theory with the apparently contradictory results of Wellisch. The essential difference between the present experiments and those of Wellisch lies in the fact that Wellisch using the *Franck modification* of the Rutherford alternating

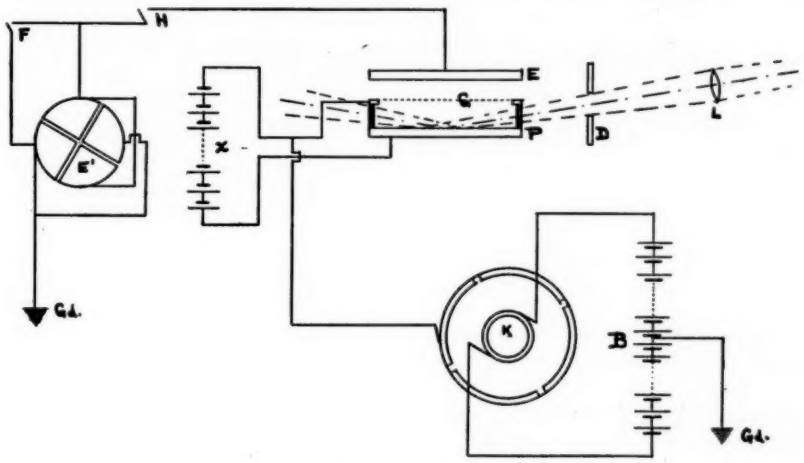


Fig. 13.

current method generated his ions by means of alpha particles in a sort of auxiliary chamber *PG*, Fig. 13. Furthermore Wellisch in general worked at pressures well below those of the other workers.

In an auxiliary chamber such as *PG* electrons liberated in the process of ionization have to travel distances as great as 2 cm. in air in fields of about 8 volts per cm. They thus make many collisions with neutral

molecules before passing through the meshes of the gauze G into the measuring field GE. Accordingly on the Thomson theory there must be quite a proportion of carriers which though starting as electrons form attachments by virtue of their many encounters and therefore enter the space GE as ions. These, in measurements of the Wellisch type, give mobilities which are perfectly normal, *i.e.*, the measured mobility should be inversely proportional to the pressure. Besides such carriers there

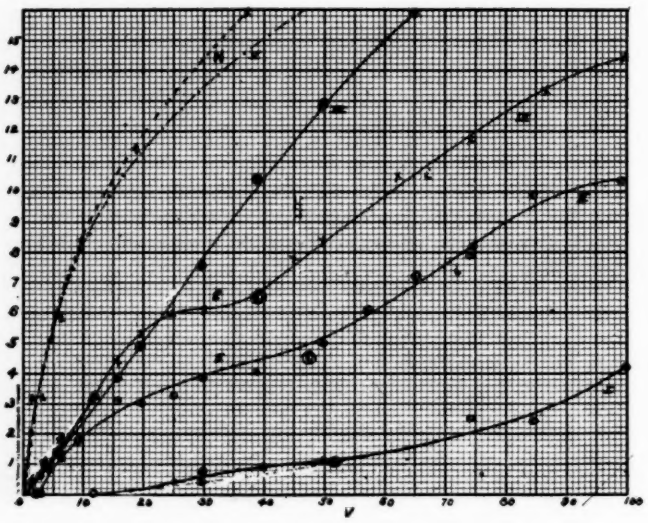


Fig. 14.

May 21, 1920. Using auxiliary gauze G 1.5 cm. above P, $d = 1.33$, $p = 49$ mm., $N = 710$.

I.	$X = 1.5$ volts	Mobility	$K = 1.65$
II.	$X = 4.5$ "	"	$K = 1.80$
III.	$X = 9.0$ "	"	$K = 2.20$
IV.	$X = 45.0$ "	"	.....
V.	$X = 45.0$ "	Saturation	—. —. —. —.

Anomalous mobility K' for all about 33.0. N curve ..... for free electrons in N_2 gas at atmospheric pressure. $N = 360$.

enter the field GE through the gauze electrons, which increase in relative numbers as the pressures decrease. These should behave just as did the photo-electrons in the experiments of the writer in giving anomalous mobilities as the Thomson theory demands. The result to be expected on applying the Thomson theory to the method employed by Wellisch would then be that the current-voltage curves obtained consist of the two parts i and E of curves II. and III., Figs. 14, 15, 16, and 17. The part due to the ions (i) should yield at its lower extremity (*i.e.*, at the point of inflection) mobilities inversely proportional to the pressure; while the other portion (E) due to electrons uniting to form ions in going

from G to E should yield the anomalous mobilities previously observed.

Now Wellisch in his experiments found curves consisting of two distinct parts which were quite similar to the curves represented above. He showed that the upper portions of his curves, corresponding to the portion of the curves marked i in Fig. 14, yielded values for the mobility-constants of the carriers which they represented equal to those of normal ions. These mobilities moreover were inversely proportional to the pressure. Wellisch however did not find that the feet of the E portions of the curves were due to carriers yielding the fictitious abnormal mobili-

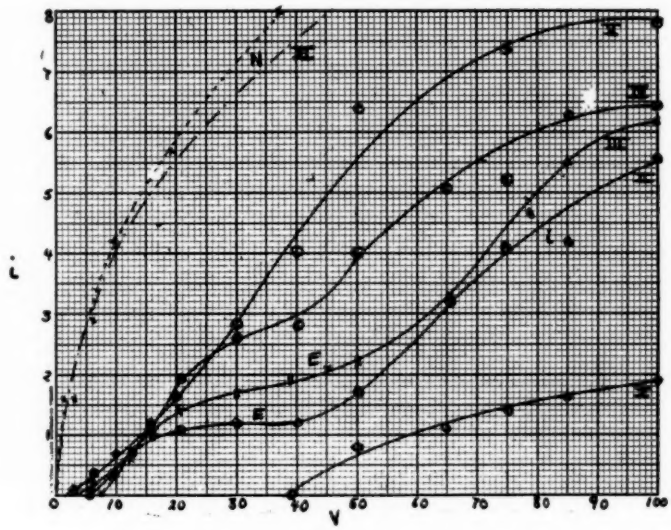


Fig. 15.

May 22, 1920. Using auxiliary gauze G 1.5 cm. above P, $d = 1.33$, $p = 56$ mm.

I.	$X = 4.5$ volts	$N = 710$	Mobility $K = 2.05$
II.	$X = 9.0$ "	$N = 710$	" $K = 1.90$
III.	$X = 13.5$ "	$N = 710$	" $K = 1.70$
IV.	$X = 22.5$ "	$N = 704$	" $K = 2.05$
V.	$X = 45.0$ "	$N = 704$	" $K = 2.20$
VI.	$X = 45.0$ "	$N = 0$	Saturation $-.-.-.-.$

Anomalous mobility K' for all curves is about 19. N curve ..... for free electrons in pure N_2 gas at atmospheric pressure and $N = 360$.

ties as required by the Thomson theory. In fact in most of his published curves the feet of his E curves cut the axis so close to the O value that he ascribed them to electronic carriers which remained *permanently free*. It was this interpretation of his curves that lead Wellisch to the theory of ion formation proposed by him.

It is then clear that to correlate the Wellisch experiments with the Thomson theory one must show that the E portions of Wellisch curves

are not due to *permanently free electrons* as Wellisch was led to believe but are due rather to the attachment, *while in the measuring field of electrons to molecules to form ions.* In consequence of the low pressures employed by Wellisch it is impossible to estimate from the intercepts of his curves with the voltage axis whether the values of the mobility-constant K' for the E type of carrier were of an electronic order of magnitude (*e.g.*, above 200 cm./sec.) (1, 7) or whether they had values of the order of magnitude which the Thomson carriers yield at those pressures.

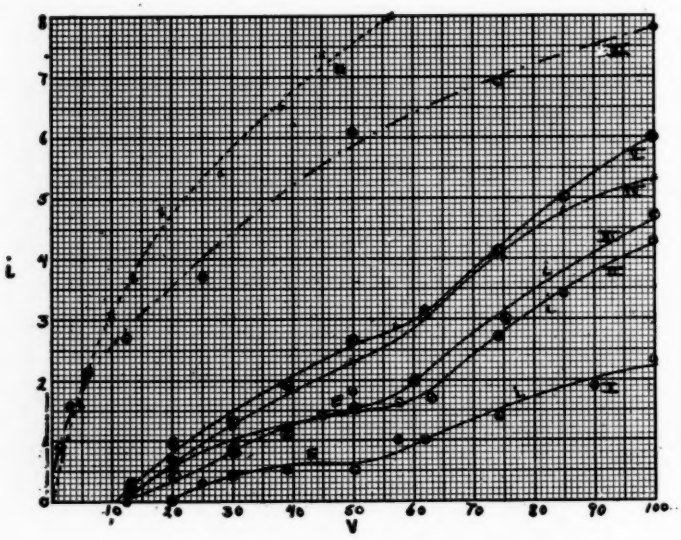


Fig. 16.

Oct. 15, 1920. Using auxiliary gauze G 1.5 cm. above P, $d = 1.28$, $p = 75$ mm.

I.	$X = 4.5$ volts	$N = 658$	Mobility $K = 1.98$
II.	$X = 9.0$ "	$N = 658$	" $K = 1.98$
III.	$X = 18.0$ "	$N = 658$	" $K = 1.98$
IV.	$X = 45.0$ "	$N = 658$	" $K = 1.98$
V.	$X = 85.5$ "	$N = 658$	" $K = 1.98$
VI.	$X = 13.5$ "	$N = 0$	Saturation $-.-.-.-.$

Anomalous mobility K' for all curves about 9.3. Curve labeled N free electron curve for N_2 gas at atmospheric pressure with $N = 147$. Curve .....

The experiments of Wellisch were accordingly repeated by the writer using pressures ranging from 34 mm. to 84 mm. (*i.e.*, where the anomalous mobilities lie between 100 and 5 cm./sec.) with a view determining from the intercepts of the E portions of the curves with the voltage axis whether the carriers yielded mobility constants K' of electronic or ionic magnitudes.

The apparatus of the writer was accordingly converted into one of the Wellisch type using photo-electrons as a source of ions. This was

accomplished by interposing the gauze G, Fig. 13, between the plate P and the plate E. The distance PG was 1.5 cm. and the distance GE was 1.30 cm. A steady potential difference X was maintained between P and G by means of a battery of small flashlight cells, x, in such a sense as to drive negative carriers liberated at P through the meshes of the gauze G. If for a fixed value of the field X a series of measurements was made of the current to the electrometer plate E, for various values of the alternating potential applied to the gauze, curves were obtained of the type found by Wellisch. A series of such curves using different values of the field X varying from 1.5 to 85.5 volts was obtained for a number of pressures ranging from 34 mm. to 84 mm. Four typical sets

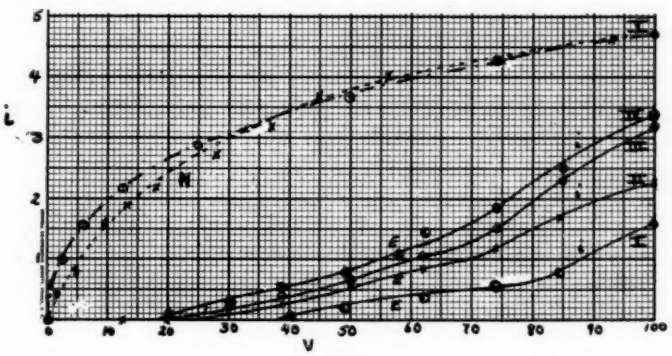


Fig. 17.

Oct. 16, 1920. Using auxiliary gauze G 1.5 cm. above P, $d = 1.28$, $p = 84$ mm.

I.	$X = 4.5$ volts	$N = 660$	Mobility $K = 1.89$
II.	$X = 13.5$ "	$N = 660$	" $K = 1.89$
III.	$X = 27.0$ "	$N = 655$	" $K = 1.89$
IV.	$X = 85.5$ "	$N = 650$	" $K = 1.89$
V.	$X = 45.0$ "	$N = 0$	Saturation −. −. −. −.

N. Curve for free electrons in N_2 gas at atmospheric pressure with $N = 146$. Curve dotted ..... Anomalous mobility K' for all curves about 6.4.

of curves are represented in Figs. 14, 15, 16, and 17 in which electrometer deflections in cm. are plotted against the alternating potential. These figures include as well the saturation photoelectric curves at those pressures, and the characteristic curve (N) yielded by purely electronic carriers (permanently free electrons), obtained in nitrogen at atmospheric pressure. The values of X as well as other data concerning the curves are given in the legend.

It is obvious at once that as the pressure increases in Figs. 14, 15, 16, and 17 the asymptotic feet of the E portions of all curves have their points of inflection (or intercepts with the V axis where points of inflection are poorly defined) at higher and higher values of the voltage, which is a characteristic of the Thomson type of carrier. It is also patent

that in no case do these portions of the curves follow the free electron curve obtained in nitrogen. If the average value of the fictitious abnormal mobility of these curves be estimated from the points of inflection, or intercepts, they yield the values 33, 19, 9.3, and 6.4 cm./sec. which are similar to those obtained for carriers in the absence of the gauze at these pressures and frequencies.

The variation in form of the Wellisch type of curves with the field X, which is obvious when one regards the set of curves for any one pressure, may be simply explained on the Thomson theory. If the values of X are low the ions spend considerable time in the auxiliary field PG before reaching the gauze G. In this time they make many impacts with neutral molecules. On the Thomson theory it is to be expected that the greater proportion of carriers passing through the gauze under these conditions would be normal ions. This is seen to be the case in curves numbered I, Figs. 14 and 15. As X increases and the time spent in PG is therefore decreased more and more electrons succeed in reaching G without attachment. The E portions of the curves consequently increase in relative importance as in curves II. and III., Figs. 14, 15, 16. On still further increasing X it is possible to cause most of the electrons to reach G without attachments. One has then precisely the conditions obtained in the absence of the gauze, curves having but one part E being observed. These are found in curves IV., Fig. 14, and V., Fig. 15. If the pressure be reduced the effect on the form of the curve is the same as is produced by increasing X, for the time spent in PG is therefore reduced.

From the foregoing one must conclude that not only is there no contradiction between the results of Wellisch, and those of the other workers, but the results of Wellisch are actually in accord with the theory of Thomson. The apparent discrepancy lay therefore only in the *interpretation* of the E portions of the Wellisch curves. Having then shown that the process of ion formation in air probably consists as Thomson postulated in the attachment of an electron to a neutral molecule which happens in one out of about 2.5×10^5 collisions, one may be permitted to speculate on the conditions determining this attachment. It has been shown by Franck (7), and also recently by the writer (10), that the electrons do not attach to form ions in pure nitrogen gas. And since the curves obtained in pure oxygen, as well as those in nitrogen contaminated with small quantities of oxygen, yield values of n of the same order of magnitude as those in air (when they are calculated back to air on the basis of their oxygen content), one may safely conclude that it is to the oxygen molecule in pure dry air that the electron attaches itself to form the negative ion. One must then picture the process of ion forma-

tion as resulting from the particular conditions surrounding the impact of an electron with an oxygen molecule, these conditions being fulfilled on the average in only one out of approximately 50,000 impacts. The nature of these conditions furnishes a problem for the future. These conditions may consist in the electrons striking a certain electronic ring in the molecule under just the right conditions of velocity. They may consist in the molecule struck being in a certain chemical or physical state, *e.g.*, ozone, or in some particular state as regards radiation, as is suggested by Perrin (11) for chemical phenomena. It is hoped that some data may shortly be available on the effect of the energy of the electron on this process. Some light will also be thrown on this process from the investigation of the behavior of other gases in this respect. Work is now under way in an attempt to determine n for different gases, and to correlate this with their chemical nature.

The writer in concluding desires to express to the National Research Fellowship Board his sincere thanks for the opportunity given him, as National Research Fellow, to clear up a problem which has fascinated him for a number of years. He also wishes to express his gratitude to Professor R. A. Millikan for the excellent facilities he placed at the writer's command for pursuing this work in the Ryerson Laboratory as well as for his kind advice and criticism in the writing of this paper.

NOTE ADDED DEC. 30, 1920.

Recent results obtained in this laboratory by Mr. Wahlin show definitely that the energy imparted to the electron is not a factor in determining its attachment to a neutral molecule to form the negative ion in air. Although Mr. Wahlin subjected electrons to fields varying from very low values up to values such that they acquired energies causing ionization by collision no noticeable increase in the formation of ions could be observed. These results definitely indicate that the theory of ion formation proposed by Wellisch is untenable.

SUMMARY.

1. Experimental work was undertaken to attempt to decide between the two theories of negative ion formation, from electrons and neutral molecules, proposed by J. J. Thomson and by Wellisch.

2. Mobilities of the carriers formed by photo-electrons liberated from one plate of a parallel plate condenser by a beam of ultraviolet light, focussed on it at a glancing angle by a quartz lens, were determined at different pressures for air using the Rutherford alternating current method.

3. The results in general confirmed the results of previous observers, yielding a single class of carriers whose mobilities became abnormal below pressures of 150 mm. The value of these mobilities was also found to be a function of the frequency of commutation in agreement with earlier results.

4. The manner of introduction of the ultra-violet light into the chamber having reduced the "stray light" effects, it was found that the asymptotic feet of the curves observed below 200 mm. pressure were a real and important feature of the phenomenon.

5. The mathematical theory of J. J. Thomson was adapted to fit these measurements; and on the basis of the equation so deduced the chance of ion formation, n, was determined from experiment.

6. Within the limits of accuracy of the method n was found to be equal to about 2.5×10^5 for pure dry air.

7. The current-voltage curves computed on the basis of the Thomson theory were compared with the observed curves and marked general similarities were noticed below 200 mm. pressures.

8. The asymptotic feet of the computed and observed curves lie close together; which is significant inasmuch as it is these portions of the observed curves that yield the abnormal values of the mobility.

9. Deviations of the observed curves from those computed at the higher and the lower pressures are explained.

10. Repetition of the Wellisch experiments shows that what he termed "free electrons" are the carriers of abnormally high mobilities observed by the earlier workers.

11. It was also shown that the identity of Wellisch "free electrons" with the carriers changing from electrons to ions in the measuring field once established, the other results of Wellisch are to be expected on the Thomson theory.

12. It is shown that as the electrons do not attach to N_2 molecules, and that as the values of n obtained in pure O_2 and in N_2 with small quantities of O_2 in it agree with the values found for air on the basis of its oxygen content, one must conclude that it is to the O_2 molecules in air that the electrons attach. The value of n for O_2 molecules is then 50,000.

REFERENCES.

1. Wellisch, E. M. Am. Jour. Science, 4, 39, p. 583, 1915; Am. Jour. Science, 4, 44, p. 1, 1917; Phil. Mag., Vol. 31, p. 186, 1916; Phil. Mag., Vol. 34, July, 1917.
2. Loeb, L. B. PHYS. REV., VIII., p. 633, 1916; Jour. Franklin Inst., p. 775, Dec., 1917.
3. Yen, Kia-Lok. PHYS. REV., Vol. 11, No. 5, p. 337, May, 1918.
4. Thomson, J. J. Phil. Mag., Sept., 1915.
5. Franck and Pohl. Verh. Deutsch. Phys. Ges., IX., p. 69, 1907.
6. Kovarik, A. F. PHYS. REV., 30, p. 415, 1910.

Tool, A. Dissertation for Ph.D. Friederich Wilhelms Institut Berlin, 1914.
Rutherford, E. Proc. Camb. Phil. Soc., 9, p. 410.
Todd. Phil. Mag., VI., 22, p. 791, 1911; Phil. Mag., June, 1913.
Townsend, J. S. Electricity in Gases, Oxford, 1914. Chap. IV.
7. Franck, J. Verh. Deutsch. Phys. Ges., 12, 613, 1910.
Haines, W. B. Phil. Mag., Oct., 1915; Phil. Mag., July, 1916.
8. Franck and Hertz. Verh. Deutsch. Phys. Ges., 15, p. 34, 1913.
9. Compton, K. T. and Benade. PHYS. REV., XI., 184.
10. Loeb, L. B. Proc. Nat. Acad. Sciences, July, 1920.
11. Perrin, Jean. Annales de Physique, Ser. 9, XI., Jan.–Feb., 1919.

THEORY AND CALCULATION OF VARIABLE ELECTRICAL SYSTEMS.

BY JOHN R. CARSON.

SYNOPSIS.

Variable Electrical Systems are defined as those in which either the circuit elements (resistance, inductance or capacity) are explicit time functions or those in which the relation between current and applied E.M.F. is non-linear. In the present paper, the *theoretical methods of solution* of "invariable systems" are extended to include variable systems by integral equations of the Volterra type. A number of representative problems are worked through to indicate the appropriate mathematical procedure and the applicability of the method to both transient and steady state phenomena.

THE symbolic or operational method of solution of problems in electric circuit theory is a highly developed and very serviceable mathematical tool which is responsible in considerable degree for the rapid developments of the more abstract side of electrical engineering where the problems encountered are of such character as to render essential an adequate theoretical guide in predicting and interpreting phenomena. The method, however, is explicitly limited in its application to those physical systems or networks which may be mathematically described by a set of linear differential equations in which the coefficients of the differential operators are constants. Physically this means two things; first the currents are proportional to the applied forces, and secondly the circuit elements of the system (resistance, inductance and capacity) are invariable.[1] A system or network in which these restrictions hold will be termed invariable. In the great majority of problems these restrictions are not serious from a practical standpoint, since the departures from the requirements of the ideal invariable systems are usually small, and can usually be taken into account by indirect methods. Theoretically, however, these restrictions are undesirable and furthermore an increasing number of problems is being encountered in which we are directly concerned with departures from the ideal requirements. Such systems, for example, are the vacuum tube where the current is not

[1] For an interesting discussion of variable electrical systems from the standpoint of dynamics, see "Theory of Variable Electrical Systems." PHYS. REV., Aug., 1917, by H. W. Nichols.

proportional to the applied electromotive force, and where the operation of the device as a detector or modulator is strictly conditioned by the fact that the characteristic is non-linear; the induction generator where the mutual inductance between primary and secondary is variable; and the microphone transmitter circuit where the resistance is varied by external means. The differential equations which describe variable systems of which the foregoing examples are typical, have, of course, been extensively studied from the standpoint of pure mathematics. At the same time curiously little application of general methods of solution appears to have been made to such physical problems, in spite of their great and increasing technical importance. In the present paper it will be shown that the differential equations of variable electrical systems can be thrown at once into the form of integral equations, by aid of which formulation the solution is quite simply expressible in terms of the solution of the corresponding invariable system. For a discussion of the application of integral equations to circuit theory the reader is referred to my paper on Transient Oscillations (March, 1919, Proc. A.I.E.E.).

The theoretical analysis of the present paper is based on the following fundamental theorem, which is derived and discussed in the paper referred to above:[1]

If the current flowing in any branch (or mesh) of a network in response to a "unit E.M.F." (zero before, unity after, time $t = 0$) is denoted by $A(t)$, then the current $I(t)$ which flows in response to the arbitrary applied E.M.F. $f(t)$ is given by the formula:

$$I(t) = \frac{d}{dt}\int_0^t f(y)A(t-y)dy. \qquad (1)$$

The function $A(t)$ which characterizes the network will be termed the *indicial admittance*.[2]

It is important to observe that formula (1) is restricted in its direct applicability to *invariable systems* as hereinbefore defined; nevertheless, as will be shown, it enables us to deal successfully with systems which are not so restricted; that is ones which contain variable circuit elements, and ones in which the relation between current and voltage is non-linear.

Since the present paper is concerned exclusively with the extension of circuit theory to systems which include variable circuit elements it is assumed in the following that the indicial admittance $A(t)$ of the invari-

[1] See also a paper by T. C. Fry on the Solution of Problems in Circuit Theory, which appeared in the PHYSICAL REVIEW, August, 1919.

[2] This terminology is suggested by the physical and mathematical significance of the function.

able or unvaried system is known. For a full discussion of the theory and calculation of invariable systems and of methods for determining the indicial admittance the reader is referred to Fry's paper, PHYSICAL REVIEW, August, 1919, and my previous papers, PHYSICAL REVIEW, September, 1917, and Proc. A.I.E.E., March, 1919.

Before developing the theory in more general terms we shall consider the simplest possible example which may be termed:

THE MICROPHONE TRANSMITTER PROBLEM.

Consider a circuit containing an invariable impedance denoted operationally by Z in series with a variable resistance $rf(t)$, to which is applied the E.M.F. $E(t)$. The equation for the resultant current $I(t)$ may be written in operational notation as:

$$(Z + rf(t))I(t) = E(t)$$

or

$$ZI(t) = E(t) - rf(t)I(t). \tag{2}$$

Inspection of equation (3) shows at once that $I(t)$ is equal to the current flowing in an *invariable circuit* of impedance Z in response to the applied E.M.F. $E(t) - rf(t)I(t)$; consequently if the *indicial admittance* of the *invariable* or *unvaried circuit* ($r = 0$) be denoted by $A(t)$ it follows at once from (1) that the current in the actual circuit may be written as:

$$I(t) = \frac{d}{dt}\int_0^t E(y)A(t-y)dy - r\frac{d}{dt}\int_0^t A(t-y)f(y)I(y)dy. \tag{3}$$

The first term on the right-hand side of (3) is by comparison with formula (1) simply the current which would flow in the unvaried circuit ($r = 0$) in response to the applied E.M.F. $E(t)$; denoting this current by $I_0(t)$ we have:

$$I(t) = I_0(t) - r\frac{d}{dt}\int_0^t A(t-y)f(y)I(y)dy. \tag{4}$$

Equation (4) is an integral equation of the Volterra type, methods for the formal solution of which are well known. The following series solution recommends itself by reason of the direct physical significance of each term of the series:

$$I(t) = I_0(t) - I_1(t) + I_2(t) - I_3(t) + \cdots + (-1)^n I_n(t), \tag{5}$$

where the successive terms of the series are defined by the relations:

$$\begin{aligned} I_1(t) &= r\frac{d}{dt}\int_0^t A(t-y)f(y)I_0(y)dy, \\ I_{n+1}(t) &= r\frac{d}{dt}\int_0^t A(t-y)f(y)I_n(y)dy. \end{aligned} \tag{6}$$

Referring to the fundamental formula (1) it will be observed that the successive terms of the series (5) as defined by (6) admit of direct physical interpretation as follows: $I_1(t)$ is equal to the current which would flow in the unvaried circuit of impedance Z in response to the fictitious applied E.M.F. $rf(t)I_0(t)$; $I_2(t)$ is equal to the current in the same circuit in response to the fictitious applied E.M.F. $rf(t)I_1(t)$; etc. That is to say the product of the variable resistance $rf(t)$ into each component current of the series (5) acts like an additional component E.M.F. in the unvaried circuit to produce an additional component current.

The solution is of course complete in that it formulates the resultant current for all types of applied forces and all possible forms of resistance variations. In particular if the impressed E.M.F. and the resistance variation are both periodic or sinusoidal the solution includes both transient as well as steady states. In this case, if we are concerned only with the ultimate steady state of the network it is not necessary to evaluate the definite integrals of (6). All that is necessary in order to write down the steady state solution corresponding to the series solution (5) is to express the product of $rf(t)$ into each component current (starting with $I_0(t)$) as a periodic time function, and then to evaluate the succeeding component current by operating on the periodic function with the impedance Z in accordance with usual operational rules. This is considered in greater detail below.

To illustrate the solution (5) and (6) the simplest possible case will be dealt with: into a circuit of unit resistance r and inductance $L = 1/a$ in which a steady current I_0 is flowing a resistance r is suddenly inserted at time $t = 0$; required the resultant current $I(t)$. In this case we have:

$$\begin{aligned} A(t) &= \text{indicial admittance of unvaried circuit} \\ &= 1 - e^{-at}, \\ f(t) &= 1, \end{aligned}$$

and the integral equation of the problem is:

$$\begin{aligned} I(t) &= I_0 - r\frac{d}{dt}\int_0^t (1 - e^{-av})I(t - y)dy \\ &= I_0 - ra\int_0^t I(t - y)e^{-av}dy. \end{aligned} \tag{7}$$

If the solution is carried out as indicated in (5) and (6), and if the notation $at = x$ is introduced, we get without difficulty

$$I(t) = I_0\left\{\begin{array}{l} 1 - r(1 - e_1(x)e^{-x}) + r^2(1 - e_2(x)e^{-x}) \\ \quad - r^3(1 - e_3(x)e^{-x}) + \end{array}\right. \tag{8}$$

Where the function $e_n(x)$ is defined as:

$$e_n(x) = 1 + x/1! + x^2/2! + x^3/3! + \cdots + x^{n-1}/(n-1)!$$
$$= \text{first } n \text{ terms of the exponential series.}$$

For all finite values of the resistance increment r the series (8) can be summed by aid of the identity

$$1 - e_n(x)e^{-x} = \int_0^x dx e^{-x} x^{n-1}/(n-1)!$$

Substitution of this identity in (8) gives

$$I(t) = I_0(1 - r\int_0^x e^{-(1+r)x}dx)$$
$$= I_0 \frac{1 + re^{-(1+r)x}}{1+r}.$$

A more interesting example than the former is presented when the applied E.M.F. and the resistance variation are both sinusoidal time functions. In this case if the frequency of the applied E.M.F. be denoted by $F = q/2\pi$ and that of the resistance variation by $f = p/2\pi$, it is easy to show that the current $I_0(t)$ in the unvaried circuit is ultimately[1] a steady state current of frequency F. This follows from the fact that the definite integral of (3) which defines the current $I_0(t)$ is resolvable into the ultimate steady state current corresponding to an applied force of frequency F, and the accompanying transient oscillations which ultimately die away. The fictitious E.M.F. which may be regarded as producing the component current $I_1(t)$ is $rf(t)I_0(t)$; this is ultimately the product of the two frequencies F and f, and therefore resolvable into two terms of frequency $F+f$ and $F-f$ respectively. Carrying through this analysis it is easy to show that each component current is ultimately a steady-state but poly-periodic oscillation, as indicated in the following table.

Component Current	Frequency	
I_0	F	
I_1	$F+f, F-f$	
I_2	$F+2f, F, F-2f$	(10)
I_3	$F+3f, F+f, F-f, F-3f$	
I_4	$F+4f, F+2f, F, F-2f, F-4f$	

It is of importance to observe that the component currents involve,

[1] It hardly seems necessary to remark that the reference time $t = 0$ is purely arbitrary and that the resistance variation may start at such a time thereafter that $I_0(t)$ may be regarded as steady state during the entire time interval in which we are interested. Going farther, if we confine our attention to sufficiently large values of t, the whole process may be treated as steady state.

from a mathematical standpoint, multiple integrals of successively higher orders, the nth component, $I_n(t)$, involving a multiple integral of the nth order with respect to $I_0(t)$. Consequently the successive currents require longer and longer intervals of time to build up to their proximate steady-state values, so that the time required for the resultant steady-state to be arrived at cannot be inferred from the time constant of the unvaried circuit.

From table (10) it will be seen that the ultimate steady-state current is obtained by rearranging the series $I_0 + I_1 + I_2$ and is of the form

$$\sum_{n=-\infty}^{+\infty} A_n \cos (q + np)t + B_n \sin (q + np)t.$$

It is interesting to note that this series comes within the definition of a Fourier series only when $q = 0$ or an exact multiple of p. The steady-state solution is of very considerable importance and is considered in more detail in a succeeding section.

So far nothing has been said regarding the convergence of the formal series solution (5). For the case of variable resistance, however, it can be shown that the sufficient conditions for the absolute convergence of the series correspond to the physical restrictions imposed in a large and important class of problems. In the first place the series is absolutely convergent when $A(o) = 0$, since in this case the successive terms are related by the equation

$$I_{n+1}(t) = r \int^t A'(t - y) f(y) I_n(y) dy,$$

where

$$A'(t - y) = \frac{d}{dt} A(t - y).$$

In physical terms this restriction means that the branch of the network in which the variable resistance is located contains inductance also. In a large number of problems this condition is satisfied.

In the second place the solution is a power series in the parameter r, which fixes the size of the resistance variation. In general therefore the series will be absolutely convergent for some restricted range of values of r, which will, however, depend on the particular network under consideration. For the very important case of periodic resistance variations physical considerations restrict the maximum value of the variable resistance to a value less than the invariable resistance in the same branch. It is easy to see from physical considerations that in this case series (5) is absolutely convergent.

However a perfectly general restriction, imposed by physical conditions,

is that $[1 + rA_{11}(0)f(t)] > 0$, since otherwise the branch would contain negative resistance. In view of this restriction an absolutely convergent series solution of equation (4) is obtainable by aid of the transformation.

$$J(t) = I(t)[(1 + rA(0)f(t)],$$

$$\phi(t) = f(t)/(1 + rA(0)f(t)).$$

In terms of J and ϕ the integral equation (4) becomes

$$J(t) = I_0(t) - r\int_0^t A'(t - y)\phi(y)J(y)dy.$$

This integral equation has the absolutely convergent solution:

$$J(t) = J_0(t) - J_1(t) + J_2(t) - \cdots,$$

where the terms of the series are defined by the relations:

$$J_0(t) = I_0(t)$$

. . . . . .

$$J_{n+1}(t) = r\int_0^t A'(t - y)\phi(y)J_n(y)dy.$$

For the sake of its physical interpretation this last equation may be written as:

$$J_{n+1}(t) = -rA(0)\phi(t)J_n(t) + r\frac{d}{dt}\int_0^t A(t - y)\phi(y)J_n(y)dy.$$

Inspection of this equation shows that the term $J_{n+1}(t)$ may be physically interpreted as the difference in the currents flowing in the unvaried network and in a resistance $1/A(0)$ in response to the fictitious E.M.F. $r\phi(t)J_n(t)$. This interpretation is of value in enabling one to write down immediately the corresponding steady-state current in the important case of periodic applied forces and periodic resistance variations.

In the light of the foregoing example the extension of the method of solution to more complicated networks and to the case of simultaneous impedance variations in a plurality of branches of the network should present no difficulties. The appropriate procedure, however, will be briefly illustrated by an example of some practical importance which may be termed:

THE INDUCTION GENERATOR PROBLEM.

In a sufficiently general form, this problem, which includes the fundamental theory of the dynamo, may be stated as follows:

Given an invariable primary and secondary circuit with a variable mutual inductance $Mf(t)$ which is an arbitrary but specified time function, and let the primary be energized by an E.M.F. $E(t)$ impressed in

the circuit at the reference time $t = 0$; required the primary and secondary currents.

In operational notation the problem may be formulated by the equations:

$$\begin{aligned} Z_{11}I_1 - pMf(t)I_2 &= E(t) \\ -pMf(t)I_1 + Z_{22}I_2 &= 0 \end{aligned} \tag{11}$$

in which Z_{11} and Z_{22} are the self impedances of the primary and secondary respectively; $Mf(t)$ is the variable mutual inductance; $E(t)$ is the applied E.M.F. in the primary, and p denotes the differential operator d/dt. By aid of the fundamental formula (1) these equations may be written down as the following simultaneous integral equations:

$$\begin{aligned} I_1(t) &= \frac{d}{dt}\int_0^t dy A_{11}(t-y)\left(E(y) + M\frac{d}{dy}[f(y)I_2(y)]\right), \\ I_2(t) &= M\frac{d}{dt}\int_0^t dy A_{22}(t-y)\frac{d}{dy}[f(y)I_1(y)]. \end{aligned} \tag{12}$$

In these equations $A_{11}(t)$ and $A_{22}(t)$ denote the indicial admittances of the primary and secondary circuits respectively (when $M = 0$); that is the currents in these circuits in response to a unit E.M.F. (zero before, unity after time $t = 0$). We of course assume that they are known or can be determined by usual methods.

It follows at once that the formal solution of these equations is the infinite series:

$$I_1(t) = X_0(t) + X_2(t) + X_4(t) + \cdots + X_{2n}(t) + \cdots \tag{13}$$

$$I_2(t) = Y_1(t) + Y_3(t) + Y_5(t) + \cdots \tag{14}$$

in which the successive terms of the series are defined as follows:

$$\begin{aligned} X_0(t) &= \frac{d}{dt}\int_0^t dy A_{11}(t-y)E(y) = I_0(t), \\ Y_1(t) &= M\frac{d}{dt}\int_0^t dy A_{22}(t-y)\frac{d}{dy}[f(y)X_0(y)], \\ X_2(t) &= M\frac{d}{dt}\int_0^t dy A_{11}(t-y)\frac{d}{dy}[f(y)Y_1(y)], \\ Y_3(t) &= M\frac{d}{dt}\int_0^t dy A_{22}(t-y)\frac{d}{dy}[f(y)X_2(y)], \qquad \text{etc.} \end{aligned} \tag{15}$$

In the light of formula (1) the physical interpretation of the series solutions (13) and (14) follows at once. Thus $X_0(t)$ is equal to the current $I_0(t)$ flowing in the *isolated* primary in response to the applied E.M.F. $E(t)$; the first component current $Y_1(t)$ in the secondary is equal to the

current which would flow in the isolated secondary in response to the applied E.M.F. $M(d/dt)f(t)X_0(t)$; $X_2(t)$, the second component current in the primary is equal to the current in the isolated primary in response to the applied E.M.F. $M(d/dt)f(t)Y_1(t)$; etc. The resultant currents are thus represented as built up by a to-and-fro interchange of energy between primary and secondary or by a series of successive reactions. In the important case where the applied E.M.F. and the variation of mutual inductance are both sinusoidal time functions of frequency F and f respectively, it is easy to show that each component current becomes ultimately equal to a set of periodic steady-state currents. Thus the component X_0 is ultimately singly periodic of frequency F; Y_1 is ultimately doubly periodic of frequencies $F+f$ and $F-f$; X_2 triply periodic of frequencies $F+2f$, F and $F-2f$; Y_3 quadruply periodic of frequencies $F=3f$, $F+f$, $F-f$, $F-3f$; etc.

In connection with this solution attention should be called to a discussion of the same problem in a paper by Liebowitz (Proc. Inst. Radio Engineers, Dec., 1915). The method of solution there employed and credited by the author to M. I. Pupin is based entirely on steady-state concepts and is limited to sinusoidal impressed forces and inductance variations. With these restrictions the solution arrived at corresponds term by term to the steady-state part of the complete solution given above. While the two solutions thus become ultimately identical in the region where Pupin's solution is convergent and valid[1] the present treatment is of broader scope in that transient as well as steady-states and arbitrary forces and inductance variations are included in the complete solution.

It is beyond the scope of the present paper to go into a consideration of the energy relations of the induction generator, but a few deductions may be noted. In the case of resistance variations the ignored force which controls the variable resistance element neither supplies nor abstracts energy; consequently in this case all the energy consumed in the system is furnished by the electrical source. In the case of the induction generator, on the other hand, the ignored force which controls the variation of mutual inductance may either supply or abstract energy from the electrical system; in other words the system may act either as an induction generator or an induction motor, and the ignored force correspondingly as a source or sink of energy. In the simple case where the series is so rapidly convergent as to make the component currents of frequencies F and $F+f$ alone of importance the system acts like an

[1] In the paper referred to, incorrect physical conclusions are deduced from the fact that the steady-state series diverges under certain conditions. As explained in the present paper no physical significance can in general be attached to the divergence of this series.

electrical generator or motor according as the frequency f of inductance variation is greater or less than the frequency F of the electrical supply. This agrees with the well-known fact that the induction motor acts like a generator when driven above synchronism. When the circuits are appropriately tuned to emphasize the higher harmonics the device is essentially a Goldschmidt generator which has been proposed for the production of radio frequency currents.

The formal series solutions are absolutely convergent in the majority of actual circuits. In special circuit arrangements, however, such as the Goldschmidt generator, where the primary and secondary are tuned to emphasize the higher harmonic currents, the series may be divergent. In this case some transformation, such as that discussed in connection with the preceding example must be introduced. For example, if the equations (12) are cleared of the differential operator, and if we introduce the functions

$$J_1(t) = I_1(t)/(1 - \mu_1\mu_2 f(t)), \qquad \mu_1 = MA_{11}'(0),$$
$$J_2(t) = I_2(t)/(1 - \mu_1\mu_2 f(t)), \qquad \mu_2 = MA_{22}'(0),$$

it is easily shown that the series solutions in J_1 and J_2 are absolutely convergent.

The Solution for the Steady-State Oscillations.

For the very important case of periodic applied forces and periodic variations of circuit elements we are often concerned exclusively with the ultimate steady-state of the system, and not at all with the mode in which the steady-state is approached; that is, attention is restricted to the periodic oscillations which the system executes after transient disturbances have died away. In this case, if the periodic variations of circuit elements are sufficiently small the required steady-state is obtained in the form of a series by replacing each term of the complete series solution by its ultimate steady-state value; a process which is very simple in view of the physical significance of each term of the latter series. The procedure will be briefly illustrated in connection with the *microphone transmitter* problem, which is formulated and solved in equations (1)–(6). The variable resistance element will be taken as $r \cos pt$, and the current $I_0(t)$ as the *real part* of $I_0 e^{iqt}$ where I_0 is in general complex, and the symbol i denotes the imaginary operator $\sqrt{-1}$. The frequency F of the impressed electric force is therefore $q/2\pi$ and the frequency f of resistance variation is $p/2\pi$. The symbolic notation commonly used in the theory of alternating current will be employed and the symbolic impedances of the unvaried network at frequency $(q + np)/2\pi$ will be

denoted by Z_n. This is obtained by methods long employed in alternating current calculations by replacing the operator d/dt by $i(q + np)$ in the differential equations which describe the network or system. Similarly the impedance of the unvaried network at frequency $(q - np)/2\pi$ will be denoted by Z_n'.

If reference is now made to equations (6) and their physical significance kept in mind it is evident at once that the component current $I_1(t)$ is equal to the current in the unvaried network in response to the applied E.M.F.

$$rf(t)I_0(t) = (r/2)I_0(e^{i(q+p)t} + e^{i(q-p)t}).$$

Consequently, after transient effects have died away, the component current $I_1(t)$ is replaceable by

$$I_1 = (r/2)I_0\left(\frac{e^{i(q+p)t}}{Z_1} + \frac{e^{i(q-p)t}}{Z_1'}\right).$$

It will be understood, of course, that the real part of this complex expression is the actual solution, and that the imaginary part is to be discarded. Proceeding in precisely the same way with the second component current $I_2(t)$, it is ultimately replaceable by

$$I_2 = (r/2)^2 I_0\left\{\frac{e^{i(q+2p)t}}{Z_1Z_2} + \frac{e^{i(q-2p)t}}{Z_1'Z_2'} + \frac{e^{iqt}}{Z_0}\left(\frac{\mathrm{I}}{Z_1} + \frac{\mathrm{I}}{Z_1'}\right)_1\right\}.$$

Similarly,

$$I_3 = (r/2)^3 I_0\left\{\begin{aligned}&\frac{e^{i(q+3p)t}}{Z_1Z_2Z_3} + \frac{e^{i(q+p)t}}{Z_1}\left(\frac{\mathrm{I}}{Z_1Z_2} + \frac{\mathrm{I}}{Z_0Z_1} + \frac{\mathrm{I}}{Z_0Z_1'}\right)\\ &+ \frac{e^{i(q-3p)t}}{Z_1'Z_2'Z_3'} + \frac{e^{i(q-p)t}}{Z_1'}\left(\frac{\mathrm{I}}{Z_1'Z_2'} + \frac{\mathrm{I}}{Z_0Z_1'} + \frac{\mathrm{I}}{Z_0Z_1}\right).\end{aligned}\right.$$

In this way the steady-state series solution is built up term by term, the component currents being poly-periodic as indicated in (10).

For sufficiently small impedance variations this method of solution works very well, and leads to a rapidly convergent solution. In other cases, however, the solution so obtained may be divergent, even when the complete series solution from which it is derived is absolutely convergent. The explanation of this lies in the fact that the steady-state series so obtained is the *sum of the limits* (as t approaches infinity) of the terms of the complete series solution, whereas the actual steady-state is the *limit of the sum*. These are not in general equal; in particular the former may be and often is divergent when the latter is convergent.

In view of the foregoing considerations it is of great importance to develop another method of investigating the steady-state oscillations which avoids the difficulties in the formal series solution. The following method has suggested itself to the writer and works very well in cases

where the previous form of solution fails. It should be stated at the outset, however, that the absolute convergence of the solution to be discussed, while reasonably certain in all physically possible systems, has not been established by a rigorous mathematical investigation which appears to present very considerable difficulties.

The method of solution will be elucidated in connection with the example discussed above under the title of the *microphone transmitter problem* and formulated in equations (1)–(6); the extension of the method of solution to more involved problems will be obvious. For the case of an applied E.M.F. of frequency $F = q/2\pi$ and a resistance variation of frequency $f = p/2\pi$ the formal series solution shows that the ultimate steady-state oscillations are of frequency F, $F \pm f$, $F \pm 2f$, $F \pm nf$. If the variable resistance is taken as $r \cos pt$ and the current $I_0(t)$ as the *real part* of $I_0 \exp(iqt)$ (where I_0 is in general complex), the following tentative solution suggests itself:

$$I(t) = A_0 e^{iqt} + \sum_{j=1}^{n} A_j e^{i(q+jp)t} + A_j' e^{i(q-jp)t} + R_n(t).$$

In this expression the coefficients A_0, A_j, A_j', which are to be determined, are in general complex and the real part of the expression is alone to be retained in the final solution. The foregoing is of course equivalent to a trigonometric series but the exponential form is much more convenient to handle. In the summation the upper limiting index n is a finite positive integer which may be assigned any desired value. The "remainder" $R_n(t)$ and the coefficients A_0, A_j, A_j' are to be determined. If the steady-state solution is convergent the remainder $R_n(t)$ must approach zero as the index n is indefinitely increased. The practical value of the solution will therefore depend on the rate of convergence of the series.

If the coefficients A_0, A_j, A_j' are determined in accordance with the process developed below, it may be shown that the remainder $R_n(t)$ for large values of t satisfies the integral equation

$$R_n(t) = -(r/2)\left(\frac{A_n}{Z_{n+1}} e^{i(q+(n+1)p)t} + \frac{A_n'}{Z'_{n+1}} e^{i(q-(n+1)p)t}\right) - r\frac{d}{dt}\int_0^t A(t-y)\cdot f(y)\cdot R_n(y)dy.$$

In this expression Z_{n+1} and Z'_{n+1} are the complex expressions for the impedance of the unvaried network at frequencies $(q + (n+1)p)/2\pi$ and $(q - (n+1)p)/2\pi$ respectively. They are obtained in accordance with well-known rules from the differential equations of the unvaried

network by replacing the differential operator d/dt by $i(q + (n + 1)p)$ and $i(q - (n + 1)p)$ respectively. The imaginary part of the foregoing expression is of course to be discarded.

The coefficients A_0, A_j, A_j' are now determined by the following set of equations which are obtained by substitution of the assumed solution in the integral equation of the problem and then letting the time t become indefinitely large.

$$\begin{aligned}
A_n &= -h_n A_{n-1}, \\
A_n' &= -h_n' A'_{n-1}, \\
A_j &= -h_j(A_{j-1} + A_{j+1}), \\
A_j' &= -h_j'(A'_{j-1} + A'_{j+1}), \qquad j = (n-1), (n-2), \cdots, 2, \\
A_1 &= -h_1(A_2 + A_0), \\
A_1' &= -h_1'(A_2' + A_0), \\
A_0 &= I_0 - h_0(A_1 + A_1').
\end{aligned}$$

In these equations the symbol h_j denotes $r/2Z_j$; similarly h_j' denotes $r/2Z_j'$.

It will be observed that starting with A_n, A_n' each coefficient is determinable in terms of the coefficient of next lower order. Thus from the first and second set of equations:

$$A_{n-1} = -h_{n-1}A_{n-2}\frac{1}{1 - h_{n-1}h_n},$$

$$A'_{n-1} = -h'_{n-1}A'_{n-2}\frac{1}{1 - h'_{n-1}h_n'}.$$

Continuing this process it is easy to show that

$$\begin{aligned}
A_j &= -h_j C_{jn} A_{j-1}, \\
A_j' &= -h_j' C'_{jn} A'_{j-1}, \qquad j = n, (n-1), \cdots, 2,
\end{aligned}$$

where C_{jn} denotes the terminating continued fraction

$$C_{jn} = \cfrac{1}{1 - h_j h_{j+1}\cfrac{1}{1 - h_{j+1}h_{j+2}\cfrac{1}{1 - h_{j+2}h_{j+3}\cdot\cdot\cdot\cfrac{1}{1 - h_{n-1}h_n}}}}$$

and C_{jn}' the corresponding expression in h_j', h_n'.

Finally, therefore

$$A_1 = -h_1 C_{1n} A_0,$$

$$A_1' = -h_1' C_{1n}' A_0$$

and

$$A_0 = I_0 - h_0(A_1 + A_1')$$

$$= I_0/(1 - h_0 h_1 C_{1n} - h_0 h_1' C_{1n}').$$

The coefficients are thus all determined in terms of I_0 and the remainder $R_n(t)$ is given by an integral equation. It follows therefore that, provided the series converges, the coefficients A_0, A_j, A_j' are the limits of the foregoing expressions as the index n is made indefinitely large and the terminating continued fractions become infinite continued fractions. The complete solution therefore involves the evaluation of infinite continued fractions.

The practical value of this method of solution will depend, of course, on the rate of convergence of the continued fractions. While no rigorous proof has been obtained, it is believed that they are absolutely convergent for all physically possible systems, but this question certainly requires fuller investigation. Nevertheless any doubt regarding the convergence of the solution need not prevent the use of the method in a great many problems where physical considerations furnish a safe guide. For example this method of solution, when applied to the problem of the induction generator, discussed above leads to the usual simplified engineering theory of the induction generator and motor, besides exhibiting effects which the usual treatment either ignores or fails to recognize.

It seems worth while pointing out that the method of solution just discussed does not exclude the investigation of the transient disturbances which exist when the electrical forces are impressed on the system or when the impedance variations are initiated. To show this in connection with the variable resistance problem, let the steady-state current, as derived above, be denoted by $S(t)$ and the total current $I(t)$ by $S(t) + T(t)$. Substitution in equation (4) gives:

$$T(t) = I_0(t) - S(t) - r\frac{d}{dt}\int_0^t A(t-y)f(y)S(y)dy - r\frac{d}{dt}\int_0^t A(t-y)f(y)T(y)dy,$$

which determines the transient disturbance $T(t)$.

NON-LINEAR CIRCUITS.

In the previous examples discussed the variations of the variable circuit elements are assumed to be specified time functions, which is the

same thing as postulating that these variations are controlled by ignored forces which do not explicitly appear in the statement and equations of the problem. We distinguish another type of variable circuit element where the variation is not an explicit time function, but rather a function of the current (and its derivatives) which is flowing through the circuit. For example the inductance of an iron-core coil varies with the current strength as a consequence of magnetic saturation. The equation of a circuit which contains such a variable element may be written down in operational notation

$$ZI + \phi(I) = E(t),$$

or

$$ZI = E(t) - \phi[I(t)]. \tag{16}$$

In this equation Z is, of course, to be taken as the impedance of the invariable part of the circuit, the indicial admittance of which is denoted by the usual symbol $A(t)$.

Equation (16) may be interpreted as the equation of the current $I(t)$ in a circuit of invariable impedance Z when subjected to an applied E.M.F. $E(t) - \phi[I(t)]$; consequently by aid of formula (1), $I(t)$ is given by

$$I(t) = \frac{d}{dt}\int_0^t A(t-y)E(y)dy - \frac{d}{dt}\int_0^t A(t-y)\phi[I(y)]dy. \tag{17}$$

The first integral is simply the current in the invariable circuit of impedance Z in response to the applied E.M.F. $E(t)$; denoting this by $I_0(t)$ we have

$$I(t) = I_0(t) - \frac{d}{dt}\int_0^t A(t-y)\phi[I(y)]dy. \tag{18}$$

This is a *functional integral equation*, the solution of which is gotten by some process of successive approximations. For example, provided the sequence converges, $I(t)$ is the limit as n approaches infinity of the *sequence*

$$I_0(t),\ I_1(t),\ I_2(t),\ \cdots,\ I_n(t), \tag{19}$$

where the successive terms of the sequence are defined by the relations:

$$\begin{aligned} I_1(t) &= I_0(t) - \frac{d}{dt}\int_0^t A(t-y)\phi[I_0(y)]dy \\ &\cdots\cdots\cdots\cdots \\ I_{n+1}(t) &= I_0(t) - \frac{d}{dt}\int_0^t A(t-y)\phi[I_n(y)]dy. \end{aligned} \tag{20}$$

With this brief sketch, the method will now be applied to *The Problem of the Three-Element Vacuum Tube.*[1]

[1] The reader is assumed in the following to be acquainted with the theory of the vacuum tube in its broad outlines. For a very able account of the theory and physics of the tube see Van der Bijl's papers, PHYSICAL REVIEW, Sept., 1918, and Proc. Inst. Radio Engineers, April, 1919. For some mathematical methods of dealing with the device see my paper, Proc. Inst. Radio Engineers, April, 1919.

In this device the output circuit or plate current is assumed to be a known function of the grid-filament and plate-filament potential differences. It is further assumed that the output or plate circuit is closed through an impedance Z (whose circuit elements are invariable) and that a specified potential difference $E(t)$ is applied between the grid and filament. We denote by $V(t)$ the potential difference between plate and filament and by $I(t)$ the unknown plate current[1] which we are to determine. We assume that $I(t)$ is a known function of $E(t)$ and $V(t)$ and therefore write

$$I(t) = F[E(t), V(t)], \tag{21}$$

But since this same current flows into the output circuit impedance Z (of indicial admittance $A(t)$) across whose terminals the potential difference is $V(t)$, we have also

$$I(t) = \frac{d}{dt}\int_0^t A(t-y)V(y)dy. \tag{22}$$

Equating (20) and (21) we get the functional equation:

$$F[E(t), V(t)] = \frac{d}{dt}\int_0^t A(t-y)V(y)dy. \tag{23}$$

Now in the actual tube the work of Van der Bijl and others has shown that over a considerable part of the characteristic the current is given by the approximate relation

$$I(t) = \frac{\mu E(t) - V(t)}{R}, \tag{24}$$

where μ is a physical parameter of the tube commonly termed the amplification factor and R is the "internal resistance" of the tube. This suggests that the characteristic function be written as:

$$F(E, V) = \frac{\mu E - V}{R} + \phi(E, V). \tag{25}$$

Whence by substitution in (23) and rearrangement we get

$$V(t) = \mu E(t) + R\phi[E(t), V(t)] - R\frac{d}{dt}\int_0^t A(t-y)V(y)dy. \tag{26}$$

Which is a functional integral equation in the unknown potential difference $V(t)$. With this function determined the current $I(t)$ is given by (20) or (21).

The solution of this functional integral equation is obtained by a process of successive approximations; in the present problem the most con-

[1] The potential differences $E(t)$, $V(t)$ and the current $I(t)$ are to be understood as denoting not the total values but rather their variations from the normal or steady condition. Thus $I(t)$ does not include the steady d.c. current.

venient and physically significant mode of approximation is to take $V(t)$ as the limit (as the index n increases indefinitely) of the sequence

$$V_0(t),\ V_1(t),\ V_2(t),\ \cdots,\ V_n(t), \tag{27}$$

where the successive approximations are defined by the equations:

$$\begin{aligned} V_0(t) &= \mu E(t) - R\frac{d}{dt}\int_0^t A(t-y)\,V_0(y)dy \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ V_{n+1}(t) &= \mu E(t) + R\phi[E(t),\ V_n(t)] - R\frac{d}{dt}\int_0^t A(t-y)\,V_{n+1}(y)dy. \end{aligned} \tag{28}$$

The physical significance of these approximations, which is of importance in the following discussion, may be seen as follows: Consider a circuit of resistance R in series with an impedance Z (of indicial admittance $A(t)$) with a potential difference $E(t)$ impressed on the terminals of the circuit, and let $V(t)$ denote the unknown voltage developed across the impedance Z. The following expressions for the current $I(t)$ in the circuit can be written down at once:

$$\begin{aligned} I(t) &= \frac{1}{R}(E(t) - V(t)) \\ &= \frac{d}{dt}\int_0^t A(t-y)\,V(y)dy. \end{aligned}$$

Equation of these two expressions gives the following integral equation for $V(t)$:

$$V(t) = E(t) - R\int_0^t A(t-y)\,V(y)dy. \tag{29}$$

Comparison of this equation with (28) shows at once that the first term $V_0(t)$ of the sequence (27) is simply equal to the voltage developed across the output circuit impedance Z on which an E.M.F. μE is impressed through a resistance R. Similarly $V_{n+1}(t)$ is equal to the potential developed across Z when an E.M.F. $\mu E + R\phi(E,\ V_n)$ is impressed thereon through a resistance R. It follows, therefore, that the resistance R is to be regarded as the "internal resistance" of the tube. In the engineering theory of the amplifier this is usually defined by the relation $1/R = \partial F(E,\ V)/\partial V$, but in the present discussion its value is not so limited. A second consequence of the above is that, so long as the characteristic is approximately a straight line the equivalent circuit of the tube is simply a resistance R in series with an impedance Z on which an E.M.F. μE is impressed. This agrees with the engineering theory of the vacuum tube amplifier[1] and shows that the simple theory is equivalent to the first approximation of the complete solution.

[1] See papers by Van der Bijl and the writer in Proc. T. R. E., April, 1919, and also a paper by H. W. Nichols, PHYSICAL REVIEW, June, 1919.

This physical interpretation is of substantial advantage in the following discussion of the convergence of the sequence. From the physics of the vacuum tube and the way in which the ϕ function is formed from the characteristic function $F(E, V)$ of the tube, it may be shown that

$$\phi(E, V) \leqq (\mu E - V)/R$$

and

$$\phi(E, V_n) - \phi(E, V_{n+1}) \leqq (V_{n+1} - V_n)/R.$$

If $V_{n+1}(t) - V_n(t)$ is denoted by $d_{n+1}(t)$, subtraction of the nth from the $(n+1)$st equation of (28) gives

$$d_{n+1}(t) + R\frac{d}{dt}\int_0^t A(t-y)d_{n+1}(y)dy \leqq d_n(t).$$

In the light of the physical interpretation of the integral equation of this type, it is immediately evident that $d_{n+1}(t)$ is related to $d_n(t)$ as the voltage across an impedance Z is to the potential impressed thereon through a resistance R. It follows therefore from physical considerations that $d_{n+1} \leqq d_n$ in all circuits of practical importance and that the sequence is convergent.

Referring to equations (28) it will be observed that each stage of the approximation formally requires the solution of an integral equation. As a matter of fact this operation can be dispensed with and the solution written as

$$V_0(t) = \frac{d}{dt}\int_0^t K(t-y)\mu E(y)dy$$

. . . . . .

$$V_{n+1}(t) = V_0(t) - R\frac{d}{dt}\int_0^t dyK(t-y)\phi[E(y), V_n(y)].$$

The function $K(t)$ may be itself evaluated from the integral equation

$$K(t) = 1 - R\frac{d}{dt}\int_0^t A(t-y)K(y)dy$$

Since this equation, physically interpreted, states that $K(t)$ is the voltage developed across the impedance Z when a "unit E.M.F." (zero before, unity after, time $t = 0$) is impressed thereon through a resistance R, it is legitimate to regard $K(t)$ as a datum of the problem, like the indicial admittance $A(t)$, since it can be evaluated by processes applicable to the theory of invariable circuits.

In view of the physical significance of the successive terms of the sequence, the actual evaluation of the integral equations and the definite integrals may be dispensed with if we are concerned only with steady-state phenomena, and the integral equations of (28) may be replaced by

equivalent operational formulæ which are calculable by usual methods. Thus if $E(t)$ is a periodic time function and we ignore transient states, we have

$$V_0 = \mu E \frac{Z}{R+Z}, \tag{30}$$

which follows at once from the physical interpretation of the sequence. V_0 is thus a periodic function of the same frequency as that of the impressed E.M.F. E. Having calculated V_0, we have in operational notation:

$$V_1 = (\mu E + R\phi(E, V_0)) \frac{Z}{R+Z}. \tag{31}$$

This can be solved for V_1 by usual steady-state methods provided that $\phi(E, V_0)$ is expanded either as a Fourier series, which is always possible, or preferably as a power series in E and V_0 which is usually possible over the operating range of the characteristic. Higher approximations $V_2, V_3, \cdots$ follow by straightforward operations.

CONCLUSION.

The purpose of the present paper has been to illustrate in a few representative problems the application of integral equations to the solution of those problems in electric circuit theory in which variable circuit elements are involved. Integral equations have been employed for some time by the writer in the solution, both formal and numerical, of practical problems in circuit theory and have proved to be a serviceable instrument. In the present paper the emphasis has been placed on general methods and the physical interpretation of the solutions, and no attempt has been made to discuss the appropriate methods of numerical solution. A considerable experience, however, has convinced the writer that the application of integral equations to the problems of circuit theory is attended by marked advantages in actual engineering calculations.

DEPARTMENT OF DEVELOPMENT AND RESEARCH,
AMERICAN TELEPHONE AND TELEGRAPH COMPANY,
March 15, 1920.

THE BLACKENING OF A PHOTOGRAPHIC PLATE AS A FUNCTION OF INTENSITY OF LIGHT AND TIME OF EXPOSURE.

BY P. S. HELMICK.

SYNOPSIS.

Blackening of Photographic Plates as a Function of Light Intensity, Wave-length and Exposure.—Three emulsions, a slow one (Seed 23), a rapid one (Seed 27*X*) and a very rapid one (Seed Graflex) were coated on plate glass and illuminated for various lengths of time with various intensities of monochromatic light of wave-length 450, 550 and 650 $\mu\mu$. The results are given in curves and are found to be well represented, for a given emulsion and wave-length, by the *empirical equation for blackening*, $D = (1/a) \log [b - (b - 1)e^{-ct}]$, where $\log c = A + B \log I + c \log^2 I$, and a, b, A, B and C are constants. A graphical method of determining these constants employing a monographic chart is described. Schwarzschild's exponent p is shown to be equal to $(B + C \log I)^{-1}$, and is found to vary from 0.7 to 1.95 according to the emulsion, wave-length and intensity. In using the *photographic method of measuring the intensity of a source of light*, it is obviously important to determine the constants of the equation for the blackening of the emulsion for the proper wave-length. The method of doing this is explained in detail.

THE object of this research has been the expression of the blackening of a photographic plate as a function of the intensity of monochromatic light and time of exposure to light, other factors being constant.

Such an expression finds practical application in photometry of monochromatic sources. By exposing a plate to a light source of unknown intensity for any known length of time, measurement of the resulting blackening will enable the unknown intensity of light to be calculated.

APPARATUS AND MATERIALS.

The plates used in the work were coated on special plate glass, and the variations of density due to unevenness of coating were of the order of 1 per cent. Three emulsions were used: "*A*" a slow emulsion (Seed 23), "*B*" a rapid emulsion (Seed 27*X*), and "*C*" a very rapid emulsion (Seed Graflex).

The source of light was a Hilger Monochromatic Illuminator. Light from a constant-voltage tungsten lamp was focussed upon the collimating slit of the illuminator, and was again focused upon the second slit of the instrument, before emerging into the exposing box.

Variations in intensity of light were obtained by varying the distance

between the plate and the second slit of the illuminator, assuming the inverse square law, and applying the necessary correction for finite width of source.[1]

Intensities were measured in watts/sq. meter by a thermopile calibrated against a standard lamp whose radiation was certified to by the Bureau of Standards.

Exposures were made by cutting one of the 5 x 7 plates into 30 strips, and exposing 12 of these, the remaining strips being used to obtain the amount of fog, as explained later. Three wave-lengths of light were used: 450 $\mu\mu$, 550 $\mu\mu$, and 650 $\mu\mu$. A hand-operated shutter was mounted in front of the slit of the illuminator, and by listening to a telephone receiver clicking seconds, exposures consisting of any integral number of seconds could easily be made.

Plates were developed at a practically constant temperature in a developer compounded after Brush's formula.[2] The maximum variation in temperature during development of the plates which make up one of the sets of curves shown later on was never more than 1°.8 C., generally it was only a few tenths of a degree. A time of development of 6 minutes at 23° C. was selected as standard. This variation in temperature was corrected for by the use of Watkins' formula[3]

$$r = (s_1/s_2)^{\frac{10}{t_2-t_1}},$$

where r = temperature coefficient, t_1 = low temperature, t_2 = high temperature, s_1 = time of development at t_1, and s_2 = time of development at t_2.[4]

The density of a photographic plate is defined as $\text{Log}_{10}\, o$, where o, the opacity, is the ratio of the incident to the transmitted light. Densities were measured in a modification of Lemon's spectro-photometer,[5] in which the prism is replaced by two mirrors inclined to one another so as to reflect two beams of light into the observing telescope. The density corresponding to an angle θ between transmitting planes of the nicols when rotated so as to cut down one light beam to the same intensity which the interposed plate of unknown density cuts down the other light beam is given by the expression $\text{Log}_{10}\, \text{Sec}^2\, \theta$.

[1] See Bull. Bur. Standards, 3, 81; 1907.

[2] PHYS. REV., 31, 243; 1910.

[3] The A. Photographer and Photo. News, London, May 17, 1910.

[4] According to Watkins, the value of "r" for any hydroquinone developer is 2.25, so this value was accordingly taken for the "r" of the hydroquinone developer used in this work. For example, if the temperature were 22° C., the time of development allowed was given by the solution of the equation

$$2.25 = \left[\frac{S_1}{6}\right]^{\frac{10}{33-22}}.$$

[5] Astrophys. Journ., 39, 204; 1914.

Densities were measured with light of 535 $\mu\mu$, obtained by placing two "monochromatic" color filters in front of the observing telescope. Density readings on a single plate were repeated until the probable error of the mean was reduced to a very small value.

Brand A.

Wave-length 550 $\mu\mu$. Distance 226.2 cm. 1.53×10^{-4} watts/sq. m. Temperature 22°.0. Time Devel. $6^m\ 30^s$.

Plate.	Time.	D	Plate.	Time.	D
1500.....		0.35	1514.....		0.29
1501.....	15^m	6.31	1515.....	15^s	0.32
1502.....		0.39	1516.....		0.26
1503.....	8^m	4.03	1517.....	8^s	0.25
1504.....		0.37	1520.....		0.25
1505.....	4^m	2.88	1521.....	4^s	0.27
1506.....		0.42	1522.....		0.27
1507.....	2^m	1.91	1523.....	2^s	0.28
1510.....		0.34	1524.....		0.27
1511.....	1^m	1.35	1525.....	1^s	0.26
1512.....		0.31	1526.....		0.24
1513.....	30^s	0.58	1527.....		0.28

The preceding table is a sample of one set of exposures. All of the separate small plates were cut from the same 5 x 7 plate. The small plates were numbered to show how they were cut with respect to the original plate. For example 1513 lay adjacent to 1512, 1514, 1503 and 1523 in the original plate. The even numbers were not exposed, and were used to obtain the fog. These unexposed fog strips were developed along with the exposed plates in order to find the blackening of the exposed plates which was not due to exposure to light. This fog density was then subtracted from the densities of the exposed plates in the calculations and curves.

The plates were of brand "*A*," exposed to light of wave-length 550 $\mu\mu$ at a distance of 226.2 cm. from the slit of the monochromatic illuminator, the flow of energy on the plate being 1.53×10^{-4} watts/sq. m. The plates were developed at a temperature of 22°.0 for a time of $6^m\ 30^s$.

The table gives the plate number, the time of exposure, and the mean of the density determinations of the plate.

Theoretical.

In 1890, Hurter and Driffield[1] developed an equation of the following form for the density D of a photographic plate exposed for a time t to light of constant intensity I:

$$D = \alpha \operatorname{Log}_e [\beta - (\beta - 1)e^{-\gamma It}],$$

[1] Journ. Soc. Chem. Ind., 9, 455; 1890.

where α, β, and γ are parameters dependent upon the character of the plate and the developer, $-\beta$ is equal to the opacity of the unexposed plate. They assumed that the amount of silver made developable by the light was proportional to $I \cdot t$, and that only the light absorbed by the unaltered silver particles contributed to the building up of the latent image.

This expression would lead to the so-called "Reciprocity Law" of Bunsen and Roscoe[1] ($I \cdot t$ equals constant for equal blackening) which Abney,[2] Miethe,[3] Eder,[4] Michalke and Schiener, Schwarzschild,[5] Kron,[6] Lemon,[7] and others have shown to be only a very rough approximation to the facts.

However, for a constant intensity of light, the expression gives a very good relationship between the density of the plate and the time of exposure, as both Hurter and Driffield, and Sheppard and Mees,[8] have shown.

In this work the equation of Hurter and Driffield is made use of in the following way:

Given a set of curves showing the relationship between density and time of exposure for intensities of light $I_1 = k_1$, $I_2 = k_2$, $\cdots$, $I_n = k_n$.

We will rewrite the equation of Hurter and Driffield in the form:

$$D = 1/a \operatorname{Log}_e [b - (b - 1)e^{-ct}].$$

Then for the curves I_1, I_2, $\cdots$, I_n, we can find values of the parameters a, b, c_1, c_2, $\cdots$, c_n, respectively which will give us this set of n curves. But c is a function of I, and if we can find this function, we can write our equation in the form:

$$D = 1/a \operatorname{Log}_e [b - (b - 1)e^{-f(I)\times t}]$$

and thus the density is completely determined in terms of the intensity of light and time of exposure.

Hurter and Driffield gave no hint as to the method of determination of the parameters a, b, and c, and Sheppard and Mees give a method applicable only when $a = 1$. Two methods are here developed which may be used to find the values of these parameters from a given experimental curve.

[1] Ann. der Phys., 117, 538; 1862.

[2] Phot. Journ., Oct., 1893.
Journ. Camera Club, 8, 46.

[3] Inaug. Diss. Gottingen, 1899.

[4] Handbuch, Bd. 2.
Jahrbuch, 1899, 457.

[5] Phot. Corr., 1899. Beitrage zur Phot. Photem. d. Gestirne. Astrophys. Journ., 11, 89; 1900.

[6] Ann. der. Phys., 41, 755; 1913.

[7] Loc. cit.

[8] Investigations on the Theory of the Photographic Process. Longmans, p. 210.

Assuming that the experimental curve is of the form given above, select four points, (t_1, D_1), (t_2, D_2), (t_3, D_3), and (t_4, D_4), so that $t_2 = t_1 + \Delta t$, $t_3 = t_1 + 2\Delta t$, $t_4 = t_1 + 3\Delta t$.

Then

$$e^{a(D_1-D_2)} = \frac{b-(b-1)e^{-ct_1}}{b-(b-1)e^{-ct_1}e^{-c\Delta t}},$$

$$e^{a(D_1-D_3)} = \frac{b-(b-1)e^{-ct_1}}{b-(b-1)e^{-ct_1}e^{-2c\Delta t}},$$

$$e^{a(D_1-D_4)} = \frac{b-(b-1)e^{-ct_1}}{b-(b-1)e^{-ct_1}e^{-3c\Delta t}},$$

$$\frac{b-(b-1)e^{-ct_1}e^{-c\Delta t}}{e^{a(D_2-D_1)}} = \frac{b-(b-1)e^{-ct_1}e^{-2c\Delta t}}{e^{a(D_3-D_1)}} = \frac{b-(b-1)e^{-ct_1}e^{-3c\Delta t}}{e^{a(D_4-D_1)}},$$

$$\frac{e^{a(D_3-D_1)} - e^{a(D_4-D_1)}}{e^{a(D_2-D_1)} - e^{a(D_3-D_1)}} = e^{-c\Delta t},$$

(1) $$c = (1/\Delta t)\,\mathrm{Log}_e \frac{e^{aD_2} - e^{aD_3}}{e^{aD_3} - e^{aD_4}},$$

(2) $$(e^{aD_1} - e^{aD_2})(e^{aD_3} - e^{aD_4}) = (e^{aD_2} - e^{aD_3})^2,$$

(3) $$b = \frac{e^{aD} - e^{-ct}}{1 - e^{-ct}}.$$

These last three equations are sufficient to determine a, b, and c.

The practical difficulties connected with this method are quite material,

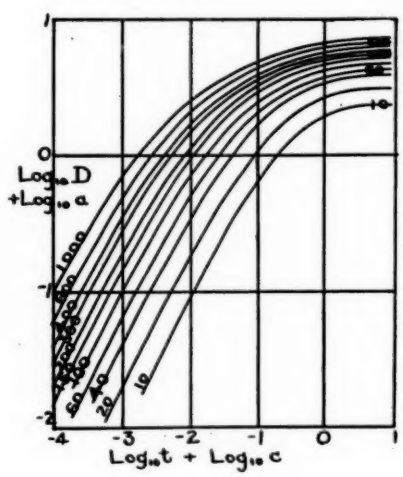


Fig. 1.

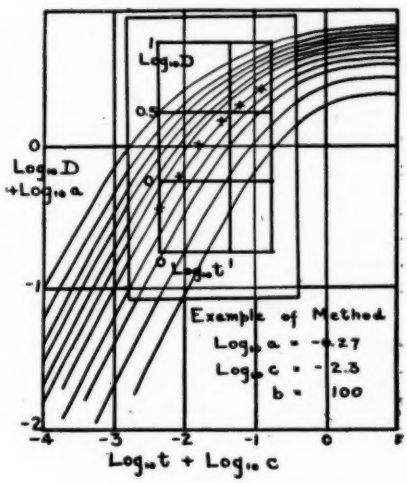


Fig. 2.

so a simple graphical method was accordingly developed.

A family of 11 curves

$$D = 1/a\ \mathrm{Log}_e\,[b - (b-1)e^{-ct}]$$

was plotted as shown in the figure, with b a parameter ranging in value from 10 to 1,000, as in Fig. 1.

Log aD was plotted along the y-axis, and Log ct along the x-axis. The experimental curve was plotted upon tracing paper with Log D and Log t as ordinate and abscissa, respectively. The tracing paper was shifted over the family of curves preserving parallelism of axes, until it coincided with one of the family. b was determined by the particular curve of the family which gave coincidence, and the a and the c were given by the intercepts of the axes of the experimental curve upon the axes of the family of curves. (Fig. 2.)

In actual practice, especially for large ranges of times of exposures, it was not difficult to find the position of the tracing paper which gave the optimum fit of the experimental curve to one of the theoretical curves,—*i.e.*, that the values of a, b, and c are unique.

Log c was next expressed as a function of Log I by Campbell's "Method of the Zero Sum,"[1] assuming that

$$\text{Log}_{10} c = A + B \text{ Log}_{10} I + C \text{ Log}_{10}^2 I.$$

This relationship was selected because the curvature of the Log c − Log I curve was very small, and because the equation readily lends itself to methods of adjustment of constants, and to the determination of the exponent "p" in Schwarzschild's "Law."[1] ($I \cdot t^p$ equals constant for equal blackening.)

In one case, namely the set of plates A–650, the curvature was more marked than in the other cases, and a term involving $\text{Log}_{10}^3 I$ was added to the expression.

The following values of the parameters were obtained by this method:

Plate.	a	b	A	B	C	D
A–450.....	0.63	200	3.99	1.95	0.10	
A–550.....	0.63	200	0.10	1.34	0.06	
A–650.....	0.40	200	−1.37	2.56	0.91	0.16
B–450.....	1.41	200	5.92	2.57	0.16	
B–550.....	1.12	200	−0.94	0.37	−0.12	
B–650.....	1.26	200	−1.00	1.38	0.11	
C–450.....	1.58	400	4.46	1.94	0.10	
C–550.....	1.12	100	0.51	1.00	−0.03	
C–650.....	1.58	200	−0.26	1.87	0.15	

$$\text{Log}_{10} c = A + B \text{ Log}_{10} I + C \text{ Log}_{10}^2 I + D \text{ Log}_{10}^3 I.$$

[1] Phil. Mag., 39, 177; 1920.

[2] Loc. cit.

In Fig. 3 the $\text{Log}_{10} c - \text{Log}_{10} I$ curves are plotted for the 9 different sets of plates. Points show values of "c" determined graphically.

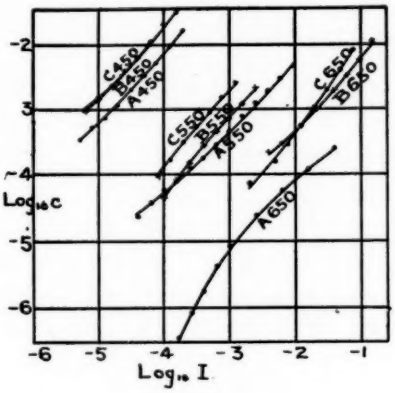


Fig. 3.

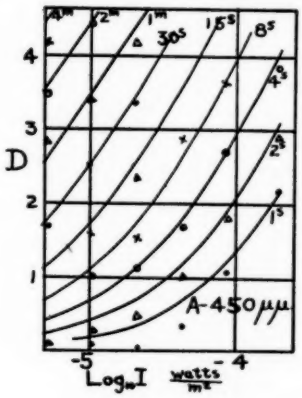


Fig. 4.

Curves show values of "c" computed from

$$\text{Log}_{10} C = A + B \text{Log}_{10} I + C \text{Log}_{10}^2 I.$$

Figs. 4 to 12 show values of blackenings computed from the above

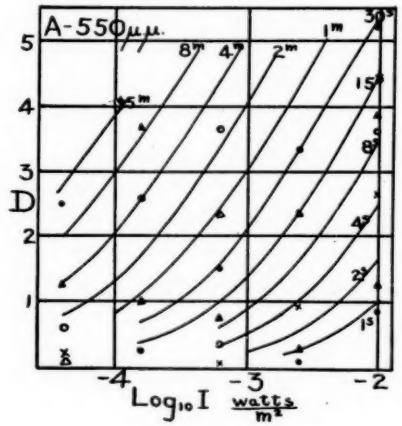


Fig. 5.

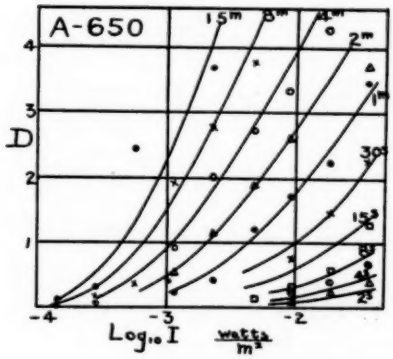


Fig. 6.

parameters, and the points represent the experimentally determined values.

The "p" of Schwarzschild's so-called "Law" can be very easily obtained from the equation which the writer has used to express the relation between the density of a plate and the intensity of light and time of exposure.

Schwarzschild states that the product $I \cdot t^p$ must be constant for equal

blackening, but Kron[1] and Lemon[2] have shown that p is a function of the intensity of light.

Expressing the blackening of a plate in the form

$$D = 1/a[\text{Log}_e b - (b - 1)e^{-ct}],$$

where

$$\text{Log}_{10} c = A + B \text{Log}_{10} I + C \text{Log}_{10}^2 I,$$

then $c \cdot t$ must be constant for equal blackening, or

$$t \cdot 10^{A+B \text{Log}_{10} I + C \text{Log}_{10}^2 I} = \text{constant.}$$

or

$$t \cdot I^{B+C \text{Log}_{10} I} = \text{constant.}$$

So $(B + C \text{Log}_{10} I)^{-1}$ is equivalent to the "p" of Schwarzschild's expression.

The following extreme values of p were computed in this manner, and show the variation in the value of p with the intensity of light.

Plate.	Range of Log_{10} Intensity.			Range of p.		
A–450.....	−3.7	−5.3		1.09	1.01	
A–550.....	−2.0	−4.4		1.22	1.06	
A–650.....	−1.4	−3.8	(−2.92 Max.)	1.59	1.36	(1.23 Max.)
B–450.....	−3.8	−5.2		1.95	1.72	
B–550.....	−2.6	−4.0		0.68	0.85	
B–650.....	−0.8	−2.4		1.30	1.13	
C–450.....	−3.8	−5.2		1.58	1.44	
C–550.....	−2.9	−4.1		1.05	1.08	
C–650.....	−1.1	−2.7		1.71	1.48	

PROCEDURE IN USING METHOD FOR PHOTOMETRIC WORK.

"Calibration" of Plates.

1. Plates are exposed to a known intensity of monochromatic light for various times of exposure and a Log D − Log t curve plotted. Values of the constants a and b are determined from this curve with the aid of the monographic chart.

2. Plates are exposed for constant times of exposure to various known intensities of the monochromatic light and a Log D − Log t curve plotted as in 1. Values of c as a function of I are now determined with the monographic chart.

3. Log c is plotted against Log I.

[1] Loc. cit.

[2] Loc. cit.

Determination of Intensity.

1. A plate is exposed for any length of time to the monochromatic light of unknown intensity.

2. Knowing the density of the plate, the time of exposure, and the

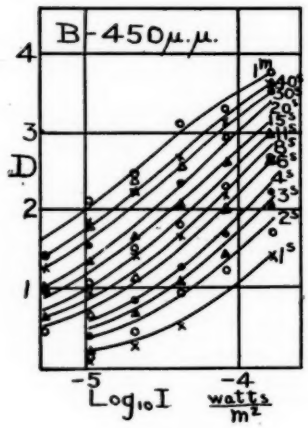


Fig. 7.

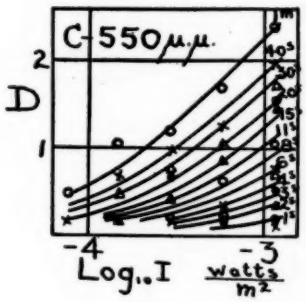


Fig. 8.

values of the constants a and b, the value of c for the particular intensity of light is determined from the monographic chart.

3. From the determined value of c, the value of I is found from the Log c − Log I curve.

Some General Considerations.

In plotting photographic blackening curves, the standard practice has always been to plot density of plate against logarithm of time of exposure, intensity being a parameter. In the light of the foregoing developments,

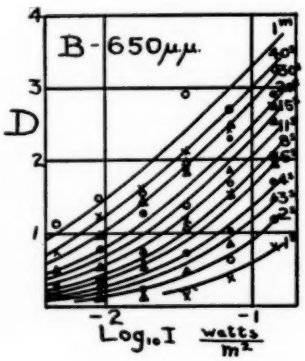


Fig. 9.

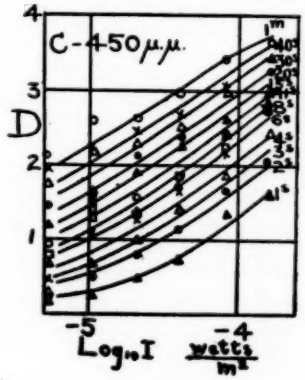


Fig. 10.

as a and b are constant for a given set of curves, it would be more logical to plot logarithm of density against logarithm of time,—for in that case the result would consist of a family of parallel curves. The actual drawing of such a family would be greatly facilitated, because the curves would be identical in shape. In this work however, the curves have

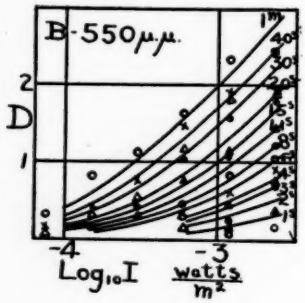


Fig. 11.

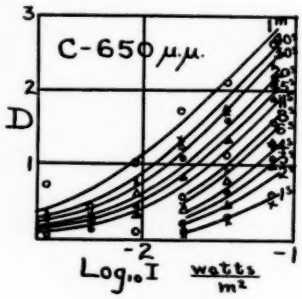


Fig. 12.

been plotted according to the standard practice in order that they may be comparable to previously published curves.

A set of exposures such as has been made throws much light upon the characteristics of fast and slow plates.

Consider first the relationship between brand of plate and blackening, with equal times of exposures and intensities of light, other factors being constant. In general, the curves of the slower plates are steeper than those of the faster ones.

The result is, that for a small intensity of light, the fast plate attains a greater density for a given time of exposure than does a slower plate. But for a longer time of exposure, the slower plate may attain a greater density than does the fast one. It is common knowledge among photographers that the longer the exposures, the less the difference that can be noticed between fast and slow plates.

The fact that the curves of the slow plates are steeper than those of the fast ones accounts for the greater contrast of a slow plate as compared to a fast one. With time of exposure constant, a small difference of intensity of light on the slow plate means a much greater difference in density in the slow plate than in the fast one.

The effect of wave-length of light upon the blackening, with intensity and time constant, is of course, very apparent, but the relation between slope of curves with wave-length of light is not marked. In fact, Leimbach[1] claims, though others have questioned his conclusions, that the slope, or "gradation" is independent of the wave-length. Upon closer

[1] Zeit. wiss. Phot., 7, 157 u. 181; 1909.

inspection, the writer's curves show a tendency to have the least gradation for that wave-length where the plate has its greatest sensibility, as Abney,[1] Stark[2] and others have found.

Certain generalizations can also be made with regard to the Log c − Log I curves. First, the curves depart more or less from a straight line. If Schwarzschild's expression were valid,—*i.e.*, $I \cdot t^p$ = constant for equal blackening, the Log c − Log I curves would be straight lines, as may be proved from the expression $p = (B - C \operatorname{Log}_{10} I)^{-1}$. The curves may therefore be regarded as giving some indication of the amount of error which is involved in assuming Schwarzschild's expression.

If Bunsen and Roscoe's "Law" held,—*i.e.*, $I \cdot t$ = constant for equal blackening, the curves would be straight lines with slope equal to unity, or c would equal I (Hurter and Driffield's assumption). The curves show that this is only a very rough approximation.

CONCLUSIONS.

Two methods of obtaining the mathematical equation of the blackening of a photographic plate as a function of intensity of monochromatic light and time of exposure have been pointed out.

Using the method of the monographic chart, the blackening equations of three different brands of plates have been calculated for three different wave-lengths of light. In these nine different cases, there is fairly good agreement between the experimental and the calculated values.

It is a pleasure to acknowledge the encouragement received from the department of physics of the State University of Iowa, and particularly from Professors L. P. Sieg, and H. L. Dodge, the latter now of the University of Oklahoma.

Ross[3] has pointed out an apparent inconsistency in the assumptions underlying the Hurter and Driffield equation made use of in the formulae above, and as a substitute develops the equation

$$D = d_m \left[1 - \frac{1}{n} \sum_{s=0}^{s=n-1} e^{-r^s k I t} \right].$$

D is the density of the plate, d_m the density for infinite exposure, I the intensity of light (assumed monochromatic, t the time of exposure, and k, r, and n are certain parameters. A method is given of determining n if d_m can be found by experiment, if arbitrary values assigned to r and k. The expression will hold only for one intensity of light unless the Reciprocity "Law" ($I \cdot t$ = constant for equal blackening) be assumed.

[1] Treatise on Photography, 10th Ed. Longmans, p. 413.

[2] Ann. der Phys., 35, 474; 1911.

[3] Journ. Optical Soc. of Am., 4, 261; September, 1920.

Ross' formula can be made the basis of expressing the blackening of a plate exposed to monochromatic light as a function of intensity of light and time of exposure according to the graphical method already outlined above if the formula is written in the form

$$D = \frac{1}{a}\left[1 - \frac{1}{n}\sum_{s=0}^{s=n-1} e^{-rct}\right],$$

and families of Log aD − Log ct curves are plotted with n and r as parameters.

A set of experimental Log D − Log t curves is obtained (I having a different value for each curve), and a particular Log aD − Log ct curve is chosen so that the n, r, and a will give a good fit to each of the Log D − Log t curves by allowing c to vary. The c and the a are found from the x and the y-intercepts of the axes of the Log D − Log t curves on the axes of the Log aD − Log ct axes, and the n and the r are determined by the particular Log aD − Log ct curve chosed to give the best fit. If c be next expressed as a function of I, we have a mathematical expression for the blackening of a photographic plate as a function of the intensity of monochromatic light and time of exposure.

The possible advantages of a graphical means of determining the parameters of Ross' equation seems to lie in the fact that by a graphical method the reader can easily judge the accuracy of fit of the experimental to the theoretical curve, and can make the curves fit particularly well at any desired point, that he can find at once the values of the parameters r, n, and k, and that he is not limited to using one particular value of r or k.

PHYSICAL LABORATORY,
THE STATE UNIVERSITY OF IOWA,
June, 1920.

SOME APPLICATIONS OF THE METHOD OF IMAGES—I.

By Jakob Kunz and P. L. Bayley.

Synopsis.

Solution of Some Electrostatic Problems by the Method of Images. (*a*) *Charged Wire between Two Parallel Plates.*—In Part I. of this paper the authors obtain expressions for the potentials at any point between an infinitely long charged wire and two conducting infinite planes parallel to it, for the surface density of the induced charge at any point on the plates, and for the capacity per unit length of such a condenser. These expressions contain only circular and hyperbolic functions. (*b*) *Charged Wire Inside an Infinite Rectangular Tube.*—In Part II., the potential at any point inside of the tube is obtained by a single infinite summation of the potentials due to each of a singly infinite set of images as given by the expression in Part I. Thus, although the problem is essentially one of doubly periodic functions, the solution appears in circular and hyperbolic functions. Comparing a square tube with a circular tube of the same capacity, each with a small wire of a given size through its center, the square tube has the larger perimeter. Tables are given showing the variation of the capacity of a certain rectangular tube with the size of the wire at its center and with the position of a certain sized wire. (*c*) *Two or Four Charged Wires Inside a Rectangular Tube.*—In the case of certain symmetrical positions this problem can be solved immediately from the preceding results.

Introduction.

GREEN'S theorem in the potential theory states that if the potential $V(x, y, z)$ in every point of a closed surface is given and also the value of ΔV in every point of the enclosed volume, then the potential V is uniquely determined in every point of that volume. This is the principle of the method of images, which we shall apply in the following cases:

I. The Capacity of an Infinite Wire between Two Infinite Parallel Planes.

Let a linear charge of e units per unit length be placed at $(a, 0)$ between the two earthed plates of infinite extent A and B of Fig. 1. Let the origin of the complex plane be taken at 0 and P be the point $z = x + iy$ at which the potential is required.

The images are shown at points z_1, z_{-1}, z_2, z_{-2}, etc., whose distances from P are r_1, r_{-1}, r_2, r_{-2}, etc. The potential at P due to the charge at a and the induced charges on the plates A and B will be calculated from the charge itself and its infinite set of images. The potential at P due

to the charge at a is $V = c - 2e \log r_0$. The resultant potential at P is then the sum of such expressions, one for each charge.

$$V_P = C + 2e[-\log r_0 + \log r_1 + \log r_{-1} - \log r_2 - \log r_{-2} + \cdots],$$

$$V_P = C + 2e \log \frac{r_1 r_{-1} r_3 r_{-3} r_5 r_{-5} \cdots}{r_0 r_2 r_{-2} r_4 r_{-4} \cdots},$$

$$V_P = C + 2e \log \frac{\prod_{n=-\infty}^{+\infty} r_{2n-1}}{\prod_{n=-\infty}^{+\infty} r_{2n}}.$$

But $r_0 = |z - z_0|$, $r_1 = |z - z_1|$, $r_{-1} = |z - z_{-1}|$, $\cdots$, $r_n = |z - z_n|$, $r_{-n} = |z - z_{-n}|$; moreover $\log r = \log |z| = R \log z$, where R denotes the real part of the logarithm. For all images of positive charge

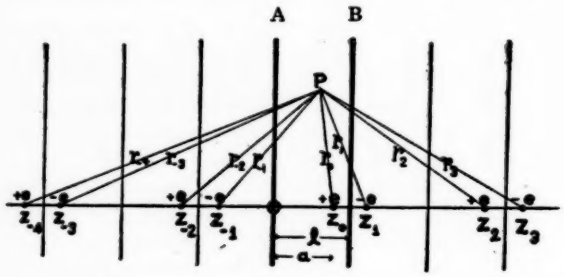


Fig. 1.

$z_{\pm 2n} = \pm 2nl + a$. For all images of negative charge $z_{\pm 2n-1} = \pm 2nl - a$. For a point taken on the plate A the potential is zero and the distances $r_0 = r_{-1}$, $r_1 = r_{-2}$, etc. Then $C = 0$; hence

$$V_P = 2e \log \frac{\Pi |z - z_{2n-1}|}{\Pi |z - z_{2n}|},$$

$$V_P = 2eR \log \frac{(z+a)(z+a-2l)(z+a+2l)(z+a-4l)\cdots}{(z-a)(z-a+2l)(z-a-2l)(z-a+4l)\cdots},$$

$$V_P = 2eR \log \frac{\dfrac{\pi}{2l}(z+a)\left[1 - \left(\dfrac{z+a}{2l}\right)^2 \dfrac{\pi^2}{\pi^2}\right] \times \left[1 - \left(\dfrac{z+a}{2l}\right)^2 \dfrac{\pi^2}{4\pi^2}\right]\left[1 - \left(\dfrac{z+a}{2l}\right)^2 \dfrac{\pi^2}{(3\pi)^2}\right] \cdots}{\dfrac{\pi}{2l}(z-a)\left[1 - \left(\dfrac{z-a}{2l}\right)^2 \dfrac{\pi^2}{\pi^2}\right] \times \left[1 - \left(\dfrac{z-a}{2l}\right)^2 \dfrac{\pi^2}{(2\pi)^2}\right]\left[1 - \left(\dfrac{z-a}{2l}\right)^2 \dfrac{\pi^3}{(3\pi)^2}\right] \cdots},$$

$$V_P = 2eR \log \frac{\sin \pi\left(\frac{z+a}{2l}\right)}{\sin \pi\left(\frac{z-a}{2l}\right)},$$

$$V_P = 2eR \log \frac{\sin \frac{\pi}{2l}(x+a)\cosh \frac{\pi}{2l}y + i \cos \frac{\pi}{2l}(x+a)\sinh \frac{\pi}{2l}y}{\sin \frac{\pi}{2l}(x-a)\cosh \frac{\pi}{2l}y + i \cos \frac{\pi}{2l}(x-a)\sinh \frac{\pi}{2l}y},$$

$$(1)\quad V_P = e \log \frac{\cosh \frac{\pi}{l}y - \cos \frac{\pi}{l}(x+a)}{\cosh \frac{\pi}{l}y - \cos \frac{\pi}{l}(x-a)}.$$

Now let us consider a wire of finite radius r having the same charge e per unit length. The potential of the surface of the wire is assumed to be the same as the potential at a point r units distant from z_0. This is equiv nt to assuming that the equipotential surface at a distance r from is a circle and is quite accurately true for small values of r when the e is not too near one of the plates. For the potential of the wire we choose

$$y = r; \qquad x = a;$$

$$V_w = e \log \frac{\cosh \frac{\pi}{l}r - \cos \frac{\pi}{l}2a}{\cosh \frac{\pi}{l}r - \cos \frac{\pi}{l}(a-a)}.$$

The capacity per unit length of the system is therefore

$$C = \frac{e}{V_w} = \frac{1}{\log \frac{\cosh \frac{\pi}{l}r - \cos \frac{\pi}{l}2a}{\cosh \frac{\pi}{l}r - 1}},$$

If $2a = l$, the capacity will be equal to:

$$C = \frac{1}{\log \frac{\cosh \frac{\pi}{l}r + 1}{\cosh \frac{\pi}{l}r - 1}}.$$

Expanding the cosh in infinite series we obtain approximately:

$$C = \frac{1}{2 \log \frac{4a}{\pi r}}.$$

From equation (1) we find the electric force in the x direction E_x as follows:

$$\frac{\partial V}{\partial x} = -E_x = e\frac{\pi}{l}\left\{\frac{\sin\frac{\pi}{l}(x+a)}{\cosh\frac{\pi}{l}y - \cos\frac{\pi}{l}(x+a)} - \frac{\sin\frac{\pi}{l}(x-a)}{\cosh\frac{\pi}{l}y - \cos\frac{\pi}{l}(x-a)}\right\}.$$

The surface density at any point on the plate A is found by

$$\sigma = -\frac{1}{4\pi}\left(\frac{\partial V}{\partial x}\right)_{x=0} = \frac{e}{2l}\frac{\sin\frac{\pi}{l}a}{\cos\frac{\pi a}{l} - \cosh\frac{\pi}{l}y}.$$

σ is a maximum for $y = 0$

$$\sigma_{y=0} = \frac{e}{2l}\frac{\sin\frac{\pi}{l}a}{\cos\frac{\pi a}{l} - 1}$$

$$= \frac{e}{2l}\operatorname{cotg}\frac{\pi a}{2l}.$$

II. The Capacity of a Circular Wire in a Rectangular Cylinder.

We shall begin with the potential due to a linear charge. Let the cylinder have a height of l cm. and breadth of l' cm. (Fig. 2). The charge

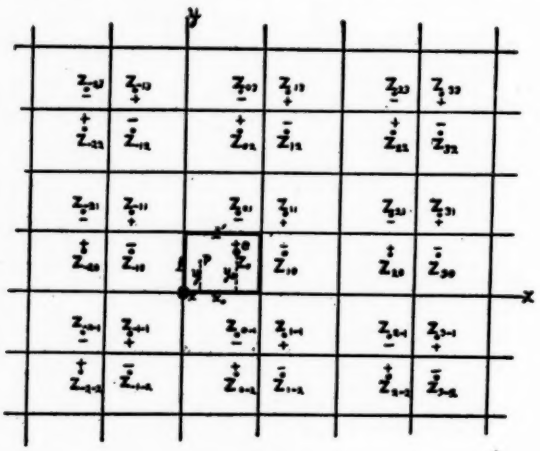


Fig. 2.

is at z_{00} or simply $z_0 = x_0 + iy_0$ and has e units per unit length. We shall remove the walls with their induced charges and consider in their stead all the infinite sets of images, which are distributed so that one of

them is in each rectangle throughout the whole complex plane. The potential at a point $P(z = x + iy)$ will first be calculated by obtaining formula (1) in generalized form applying to any row of images and then taking a single summation for all rows.

The expression for the potential due to any row of images is given by formula (1) when y is replaced by the perpendicular distance from the point P to the row of images. In Fig. 2 such a distance is represented for any row by one of the expressions:

$$(a)\quad y - y_0 + 2nl,$$

$$(b)\quad y + y_0 + 2nl,$$

where n is any positive or negative integer. The distances $x + a$, $x - a$, and l in Fig. 1 become $x + x_0$, $x - x_0$, and l' when applied to Fig. 2. The potential at P is the algebraic sum of the potentials due to all such rows of images. Noting that the potentials due to the rows whose distances are given by (b) are negative, we have

$$(2)\quad V_P = e \sum_{n=-\infty}^{+\infty} \log \frac{\left[\cosh \frac{\pi}{l'}(2nl + y - y_0) - \cos \frac{\pi}{l'}(x + x_0)\right] \times \left[\cosh \frac{\pi}{l'}(2nl + y + y_0) - \cos \frac{\pi}{l'}(x - x_0)\right]}{\left[\cosh \frac{\pi}{l'}(2nl + y - y_0) - \cos \frac{\pi}{l'}(x - x_0)\right] \times \left[\cosh \frac{\pi}{l'}(2nl + y + y_0) - \cos \frac{\pi}{l'}(x + x_0)\right]}.$$

This problem is essentially that of the conformal transformation of the rectangle with a singular point within of character $\log 1/r$ and is treated in many texts on elliptic functions. See Greenhill, Elliptic Functions, §§ 273–275; and Kneser, Die Integral Gleichungen und ihre Anwendung in der Math. Physik, p. 137. The usual solution of this problem in elliptic functions treats the whole set of images at once and is expressible in sigma or theta functions. The same expressions may be derived as follows by associating with each image a factor of a sigma function.

The coördinates of the images (Fig. 2) are given as follows: for positive images

$$z_{\pm 2m,\ \pm 2n} = z_0 \pm 2ml' \pm i2nl,$$

$$z_{\pm 2m-1,\ \pm 2n-1} = -z_0 \pm 2ml' \pm i2nl,$$

where $z_0 = x_0 + iy_0$ and $\bar{z}_0 = x_0 - iy_0$; for negative images

$$z_{\pm 2m,\ \pm 2n-1} = \bar{z}_0 \pm 2ml' \pm i2nl,$$

$$z_{\pm 2m-1,\ \pm 2n} = -\bar{z}_0 \pm 2ml' \pm i2nl,$$

then

$$V_P = 2eR \log \prod_{\substack{m=-\infty \\ n=-\infty}}^{+\infty} \frac{(z + \bar{z}_0 + \Omega)(z - \bar{z}_0 + \Omega)}{(z + z_0 + \Omega)(z - z_0 + \Omega)}, \tag{2a}$$

where $\Omega = 2ml' + i2nl$. Dividing each term by Ω and multiplying and dividing by a proper exponential factor, we may associate each term with a sigma function

$$\sigma(z) = z\Pi' \left(1 + \frac{z}{\Omega}\right) \epsilon^{z/\Omega + \frac{1}{2}(z/\Omega)^2}.$$

Reduction leaves

$$V_P = 2e \log \frac{\sigma(z + \bar{z}_0)\sigma(z - \bar{z}_0)}{\sigma(z + z_0)\sigma(z - z_0)},$$

a well-known formula which with proper interpretation has been applied to vertex motion and the flow of heat.

The sigma functions may be expanded into a single convergent product as follows:

$$\sigma(z) = \epsilon^{\eta_1 z^2/2\omega_1} \frac{2\omega_1}{\pi} \sin \frac{\pi z}{\omega_1} \prod_{n=1}^{\infty} \frac{\sin \pi\left(\frac{2n\omega_2 - z}{2\omega_1}\right) \sin \pi\left(\frac{2n\omega_2 + z}{2\omega_1}\right)}{\sin^2 \pi n \frac{\omega_2}{\omega_1}},$$

where

$$\eta_1 = \frac{\pi^2}{2\omega_1}\left[\frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{\sin^2\left(i\pi n \frac{\omega_2}{\omega_1}\right)}\right],$$

$\omega_1 = l'$, and $\omega_2 = il$. After considerable reduction we get identically equation (2).

It was suggested to the authors that for purposes of computation it is sometimes easier to use theta functions instead of the sigma function. The functions are connected in the following way:

$$\sigma(z) = A\, \epsilon^{2\eta_1 \omega \left(\frac{z}{2\omega_1}\right)^2} \vartheta_1\left(\frac{z}{2\omega_1}\right).$$

Hence,

$$\frac{\sigma(z + \bar{z}_0)\sigma(z - \bar{z}_0)}{\sigma(z + z_0)\sigma(z - z_0)}$$

$$= \epsilon^{2\eta_1\omega_1\left[\left(\frac{z+\bar{z}_0}{2\omega_1}\right)^2 + \left(\frac{z-\bar{z}_0}{2\omega_1}\right)^2 - \left(\frac{z+z_0}{2\omega_1}\right)^2 - \left(\frac{z-z_0}{2\omega_1}\right)^2\right]} \frac{\vartheta_1\left(\frac{z + \bar{z}_0}{2\omega_1}\right)\vartheta_1\left(\frac{z - \bar{z}_0}{2\omega_1}\right)}{\vartheta_1\left(\frac{z + z_0}{2\omega_1}\right)\vartheta_1\left(\frac{z - z_0}{2\omega_1}\right)}$$

$$= \epsilon^{\frac{-4\eta_1}{\omega_1} i x_0 y_0} \frac{\vartheta_1\left(\frac{z + \bar{z}_0}{2\omega_1}\right)\vartheta_1\left(\frac{z - \bar{z}_0}{2\omega_1}\right)}{\vartheta_1\left(\frac{z + z_0}{2\omega_1}\right)\vartheta_1\left(\frac{z - z_0}{2\omega_1}\right)}.$$

Substituting this expression in equation (2) and using the classical expansion of the theta functions we find essentially the same result as before.

Now let us replace the linear charge by a wire of finite radius r. If the surface of the wire coincided with an equipotential surface, then the potential could be calculated exactly from formula (2). This is very nearly the case when the dimensions l and l' are large compared with r and when the wire is not too near the walls of the cylinder. At the point P of Fig. 3 we have:

Fig. 3.

$$x + x_0 = 2x_0,$$
$$x - x_0 = 0,$$
$$y - y_0 = -r,$$
$$y + y_0 = 2y_0 - r.$$

Then

$$(3)\quad V_w = e\sum_{-\infty}^{+\infty}\log\frac{\left[\cosh\frac{\pi}{l'}(2nl-r)-\cos\frac{\pi}{l'}2x_0\right]\times\left[\cosh\frac{\pi}{l'}(2nl+2y_0-r)-1\right]}{\left[\cosh\frac{\pi}{l'}(2nl-r)-1\right]\times\left[\cosh\frac{\pi}{l'}(2nl+2y_0-r)-\cos\frac{\pi}{l'}2x_0\right]}.$$

By definition the capacity C of the system per unit length is equal to

$$C = \frac{e}{V_w}.$$

Instead of (3) we may write:

$$(4)\quad V_w = e\left\{\begin{aligned}&\sum_{n=0}^{\infty}\log\frac{\cosh\frac{\pi}{l'}(2nl+r)-\cos\frac{\pi}{l'}2x_0}{\cosh\frac{\pi}{l'}(2nl+r)-1}\\&+\sum_{n=1}^{\infty}\log\frac{\cosh\frac{\pi}{l'}(2nl-r)-\cos\frac{\pi}{l'}2x_0}{\cosh\frac{\pi}{l'}(2nl-r)-1}\\&+\sum_{n=0}^{\infty}\log\frac{\cosh\frac{\pi}{l'}(2nl-2y_0+r)-1}{\cosh\frac{\pi}{l'}(2nl-2y_0+r)-\cos\frac{\pi}{l'}2x_0}\\&+\sum_{n=1}^{\infty}\log\frac{\cosh\frac{\pi}{l'}(2nl+2y_0-r)-1}{\cosh\frac{\pi}{l'}(2nl+2y_0-r)-\cos\frac{\pi}{l'}2x_0}.\end{aligned}\right.$$

Calculations from formulæ (1) to (5) are made simple by use of the Smithsonian Institute Tables of Hyperbolic Functions. Where more than three or four significant figures are desired, the hyperbolic functions can be built up by means of the tables of exponentials in the same book.

We shall next compare the relative dimensions of circular and square cylindrical condensers of the same capacity. The capacity per unit length of a condenser formed of two concentric cylinders is given by

$$C = \frac{1}{2 \log \frac{R}{r}},$$

where R is the radius of the outer cylinder and r that of the inner cylinder. Now let the outer cylinder have a square cross-section and the inner cylinder be a wire of the same radius r as before. In the square cylinder $l = l' = 2x_0 = 2y_0$. If r/l is small in comparison to unity, it may be dropped from each term of equation (4) except in the denominator of the first term when $n = 0$. We obtain:

$$(5)\quad V_0 = e \left\{ \begin{array}{l} \log 2 - \log\left(\cosh \frac{\pi r}{l} - 1\right) + 2 \log \frac{\cosh \pi - 1}{\cosh \pi + 1} \\ + 2 \sum_{n=1}^{\infty} \log \frac{\cosh 2\pi n + 1}{\cosh 2\pi n - 1} + 2 \sum_{n=1}^{\infty} \log \frac{\cosh (2n-1)\pi - 1}{\cosh (2n-1)\pi + 1} \end{array} \right\}.$$

All the terms except the second are purely numerical quantities, which may be represented by $\log N$, hence

$$V_0 = e \log \frac{N}{\cosh \frac{\pi r}{l} - 1}.$$

The capacities of the square and circular condensers are equal if

$$2 \log \frac{R}{r} = \log \frac{N}{\cosh \frac{\pi r}{l} - 1}$$

$$= \log \frac{N}{\frac{1}{2!}\left(\frac{\pi r}{l}\right)^2 + \frac{1}{4!}\left(\frac{\pi r}{l}\right)^4 + \cdots}.$$

Neglecting powers above the second:

$$\frac{R^2}{r^2} = \frac{2N}{\left(\frac{\pi r}{l}\right)^2} \quad \text{or} \quad R = \frac{\sqrt{2N}}{\pi} l,$$

$$N = 1.43555 \quad \text{and} \quad R = 0.539364 l.$$

The circular cylinder lies partly outside the square cylinder. The perimeter of the circular cylinder is $2\pi R = 3.36l$, which is smaller than that of the square cylinder $4l$. This might have been expected from the fact that the average distance from the center to all points on the square is less than the radius of a circle of the same perimeter.

If we take a square of diagonal 2 cm. and a wire in the center of radius $r = 0.001$ cm. the formulæ (4) and (5) give the same result to within 0.2 per cent.

By means of formula (3) or (4) the capacity of a square outer cylinder of side $l = l' = 2\pi$ was calculated per unit length for different sizes of wire placed at the center. These values are given in Table I.

TABLE I.

r in cm.	0.0005	.001	.002	.01	.02	.06	.1
C in E.S.U.	.300	.326	.3565	.455	.518	.657	.755

Table II. contains the values of the capacity of a rectangular cylinder π cm. high ($l' = \pi$) 2π cm. wide ($l = 2\pi$) with a wire of radius $r = 0.01$ cm. half way between the upper and lower walls. Values of C are given for different positions of the wire as it is moved toward the side wall. The capacity changes but little for positions of the wire near the center.

TABLE II.

$y_0 = \pi/2$; $l = 2\pi$, $l' = \pi$, $r = .01$ cm.

x_0 in cm.	π	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{19}{12}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$
C in E.S.U.	.501	.503	.508	.514	.523	.580

The electric force in the x direction $E_x = -(\partial V_P/\partial x)$ in any point P, and the surface density of the induced charge in any point of the side wall ($x = 0$) is equal to

$$\sigma = -\frac{1}{4\pi}\left(\frac{\partial V}{\partial x}\right)_{x=0}.$$

From (2) we deduce:

$$-E_x = \frac{\partial V_P}{\partial x} = e\frac{\pi}{l'}\sum_{n=-\infty}^{+\infty}\left\{\frac{\sin\frac{\pi}{l'}(x+x_0)}{\cosh\frac{\pi}{l'}(2nl+y-y_0)-\cos\frac{\pi}{l'}(x+x_0)}\right.$$

$$\left.-\frac{\sin\frac{\pi}{l'}(x-x_0)}{\cosh\frac{\pi}{l'}(2nl+y-y_0)-\cos\frac{\pi}{l'}(x-x_0)}\right.$$

$$+\frac{\sin\frac{\pi}{l'}(x-x_0)}{\cosh\frac{\pi}{l'}(2nl+y+y_0)-\cos\frac{\pi}{l'}(x-x_0)}$$

$$-\frac{\sin\frac{\pi}{l'}(x+x_0)}{\cosh\frac{\pi}{l'}(2nl+y+y_0)-\cos\frac{\pi}{l'}(x+x_0)}\Bigg\}.$$

We shall finally apply the previous results to the two and four wire cable in rectangular conduits. From the distribution of images in Fig. 2 it is evident at once, that the capacity of a system represented by Fig. 4 can be found as follows: Let the charges per unit length be $+e$ and $-e$ and placed at equal distances from the walls AF and CD and on a line

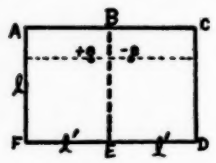


Fig. 4.

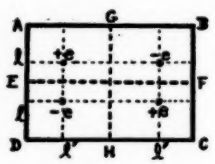


Fig. 5.

parallel to AC. The distribution of the field is not changed by inserting a conducting partition BE. This plate would become charged positively on the right-hand side and negatively on the left-hand side and the system appears as two equal condensers connected in series. Then if equation (3) is written in the form $V_w = ef(x_0, y_0, l, l', r)$, the capacity of $ABEF$ is equal to $(e/V_w) = (1/f)$ and the capacity of $ACDF$ containing both wires is equal to $1/2f$.

If four wires are placed symmetrically as shown in Fig. 5, two walls, EF and GH, may be inserted without disturbing the original field of force. We have two sets of condensers joined in series, each set consisting of two condensers in parallel. This combination has the capacity of any single condenser.

URBANA, ILLINOIS,
June 3, 1920.

SOME APPLICATIONS OF THE METHOD OF IMAGES—II.

By C. M. Hebbert.

Synopsis.

Distribution of Potential around a Charged Wire within a Rectangular Tube as Determined by the Method of Images.—The author derives the general equation for potential, given by Kunz and Bayley in the preceding paper, directly from Euler's infinite product expansion for the sine, thus avoiding the use of elliptic functions. Equipotential lines and lines of force are shown for two cases, a square and a rectangular tube; also curves giving the fall of potential along certain important lines.

At Dr. Kunz's suggestion I have carried out the calculations of potentials given by equation[1] (2) for the special cases in which the wire is at the center of a square cylinder and at the point $(l/2, l/10)$ in a rectangular cylinder of dimensions $2l \times l$. Tables and curves are given for these cases.

I have also applied Euler's infinite product expansion for the sine, as used in deriving equation (1), to the derivation of the general equation (2). This method avoids the use of the elliptic functions and arrives at the result in a direct way.

Equation (2a) may be written (taking out value for $m = 0$):

$$V_P = 2eR \log \prod_{\substack{m=-\infty \\ n=-\infty}}^{\infty} \frac{\begin{array}{l}(z + \bar{z}_0 + 2inl)(z - \bar{z}_0 + 2inl) \\ \times (z + 2ml' + 2inl + \bar{z}_0)(z + 2ml' + 2inl - \bar{z}_0)\end{array}}{\begin{array}{l}(z + z_0 + 2inl)(z - z_0 + 2inl) \\ \times (z + 2ml' + 2inl + z_0)(z + 2ml' + 2inl - z_0)\end{array}},$$

where $m \neq 0$

$$= 2eR \sum_{n=-\infty}^{\infty} \log \prod_{m=1}^{\infty} \frac{\begin{array}{l}(z + \bar{z}_0 + 2inl)(z - \bar{z}_0 + 2inl) \\ \times (z + 2ml' + 2inl + \bar{z}_0)(z - 2ml' + 2inl + \bar{z}_0) \\ \times (z + 2ml' + 2inl - \bar{z}_0)(z - 2ml' + 2inl - \bar{z}_0)\end{array}}{\begin{array}{l}(z + z_0 + 2inl)(z - z_0 + 2inl) \\ \times (z + 2ml' + 2inl + z_0)(z - 2ml' + 2inl + z_0) \\ \times (z + 2ml' + 2inl - z_0)(z - 2ml' + 2inl - z_0)\end{array}}$$

[1] The numbers refer to equations in the preceding paper.

$$= 2eR \sum_{n=-\infty}^{\infty} \log \prod_{m=1}^{\infty} \frac{\dfrac{\pi(z+\bar{z}_0+2inl)}{2l'}\left[1-\left(\dfrac{z+\bar{z}_0+2inl}{2ml'}\right)^2\right] \times \dfrac{\pi(z-\bar{z}_0+2inl)}{2l'}\left[1-\left(\dfrac{z-\bar{z}_0+2inl}{2ml'}\right)^2\right]}{\dfrac{\pi(z+z_0+2inl)}{2l'}\left[1-\left(\dfrac{z+z_0+2inl}{2ml'}\right)^2\right] \times \dfrac{\pi(z-z_0+2inl)}{2l'}\left[1-\left(\dfrac{z-z_0+2inl}{2ml'}\right)^2\right]}$$

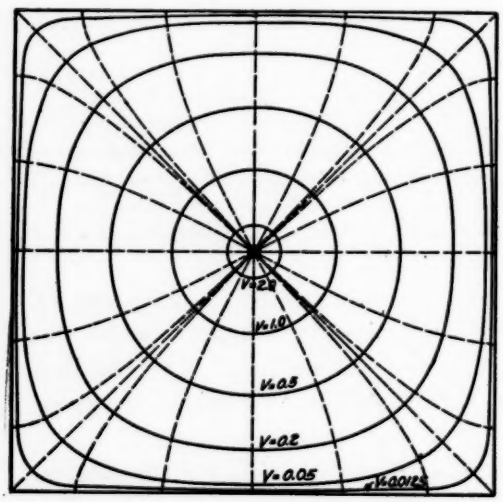


Fig. 1.

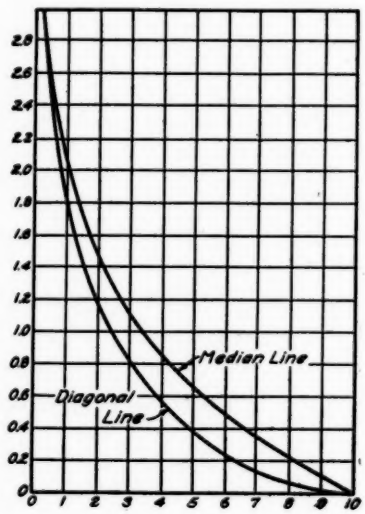


Fig. 2.

FIG. 1. Shows the equipotential curves and lines of force for a wire at the center of a square cylinder. The coördinates were read directly from the curves in Fig. 2.

FIG. 2. Shows the fall of potential along the median and diagonal lines of the square in Fig. 1. The horizontal unit represents tenths of median and diagonal respectively, measured from the center.

$$= 2eR \sum_{n=-\infty}^{\infty} \log \frac{\sin\dfrac{\pi(z+\bar{z}_0+2inl)}{2l'} \cdot \sin\dfrac{\pi(z-\bar{z}_0+2inl)}{2l'}}{\sin\dfrac{\pi(z+z_0+2inl)}{2l'} \cdot \sin\dfrac{\pi(z-z_0+2inl)}{2l'}}$$

$$= e \sum_{n=-\infty}^{\infty} \log \frac{\left[\cosh\dfrac{\pi}{l'}(y+2nl-y_0)-\cos\dfrac{\pi}{l'}(x+x_0)\right] \times \left[\cosh\dfrac{\pi}{l'}(y+2nl+y_0)-\cos\dfrac{\pi}{l'}(x-x_0)\right]}{\left[\cosh\dfrac{\pi}{l'}(y+2nl+y_0)-\cos\dfrac{\pi}{l'}(x+x_0)\right] \times \left[\cosh\dfrac{\pi}{l'}(y+2nl-y_0)-\cos\dfrac{\pi}{l'}(x-x_0)\right]}$$

which is equation (2).

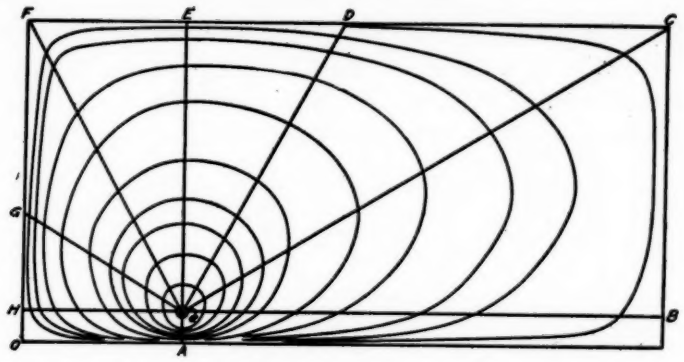
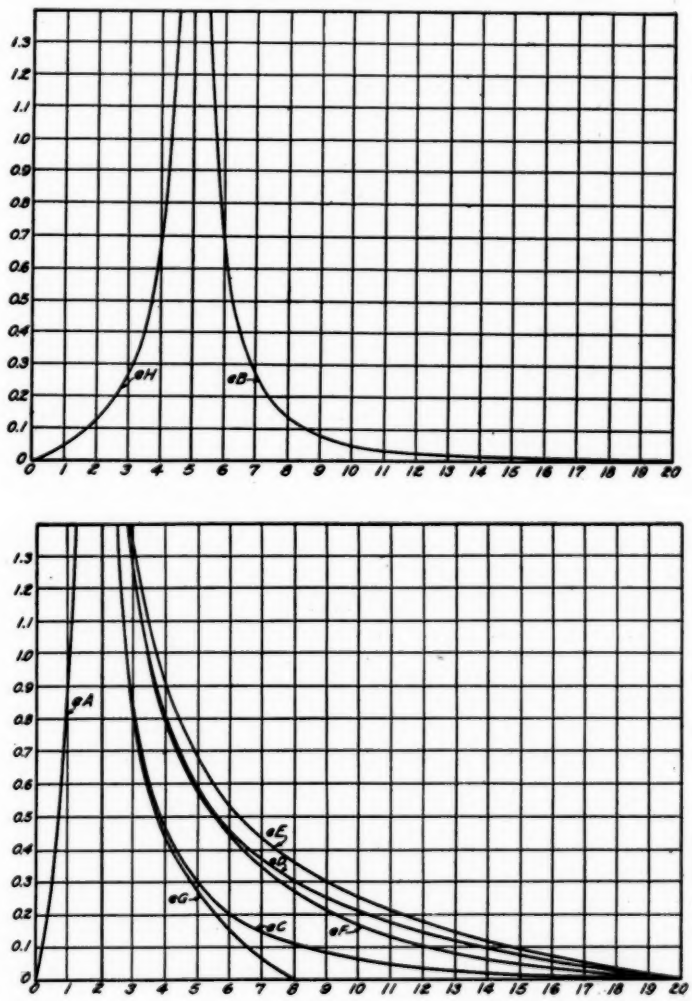


FIG. 3. Same as Fig. 1 for wire at $l/2$, $l/10$ in rectangular cylinder $2l \times l$.



FIGS. 4 and 5 Show potential drop along lines eA, eB, eC, etc. of Fig. 3. Horizontal unit in Fig. 4 is $l/10$ along x-axis and in Fig. 5 is $l/20$ along y-axis of Fig. 3.

Below are tables of calculated values for a few points. There may be errors in these values but the smoothness of the curves obtained indicates that the errors are not serious. The first six terms of the series gave sufficiently accurate results for the square, and ten terms for the rectangle.

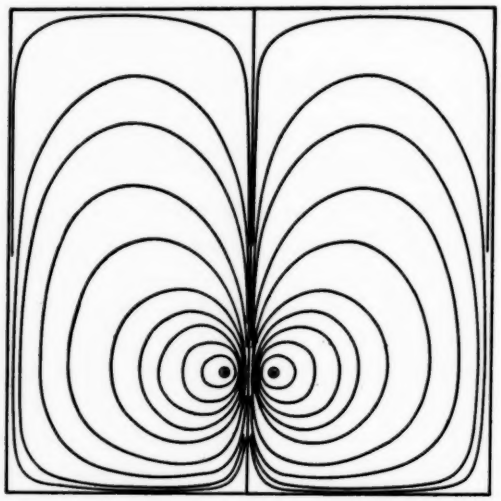


Fig. 6.

Shows the curves of Fig. 3 and their images with respect to the wall OA. The coördinates of the curves in Figs. 3 and 6 were read from the curves of Figs. 4 and 5.

The curves given in Figs. 1 to 6 are based on calculations for a large number of points not given in the tables.

TABLE I.

Calculated Potentials for Square Cylinder with e = 1 at Center. (*See Figs.* 1 *and* 2.)

Distance from center in tenths of line given	1	3	5	7	9
Median V =	2.066	1.111	.664	.360	.115
Diagonal V =	1.765	.813	.383	.135	.015

TABLE II.

Calculated Potentials for Rectangular Cylinder $2l \times l$ *with* $e = 1$ *at* $(l/2, l/10)$. (*See Figs.* 3–6.)

k =	1	2	3	4	5	6	8	10	14	15	17	18
Point $(l/2, kl/20)$, V =	.943	∞	1.366	.911	.683	.539	.362	.254	.123			.037
Point $(kl/10, l/10)$, V =	.051	.123		.674	∞	.680	.143	.050		.007	.003	

URBANA, ILLINOIS,
June 5, 1920.

THE ELECTRICAL RESISTANCE OF METALS.

BY P. W. BRIDGMAN.

SYNOPSIS.

Electrical Resistance of Eighteen Elements.—The paper contains a brief summary of an extensive series of measurements which are to be published in detail elsewhere made to determine the *effect of pressures up to* 12000 *kg. per sq. cm. and of temperatures from* 0° *to* 275° *C.* on the resistance of *lithium, sodium, potassium, gallium, bismuth, mercury, calcium, strontium, magnesium, titanium, zirconium, arsenic, tungsten, lanthanum, neodymium, carbon (amorphous and graphitic), silicon, and black phosphorus.* The data for tungsten and magnesium are improvements on data previously published; the data for the other substances are new. The first six of these elements were studied in both the liquid and the solid states. The pressure coefficients of solid calcium, solid strontium, and both solid and liquid lithium are positive; the coefficient of bismuth is positive in the solid state, but negative in the liquid.

Modified Electron Theory of Metallic Conduction.—A previous theoretical discussion of measurements of the effect of pressure on resistance suggested most strongly that in metallic conduction the electrons pass through the substance of the atoms, and that the mechanism by which resistance is produced is *intimately connected with the amplitude of atomic vibration.* This view is here given quantitative form. The classical expression for conductivity, $\sigma = (e^2/2m)(nl/v)$, is retained; the number of free electrons is supposed to remain constant, their velocity is taken to be that of a gas particle of the same mass and temperature, and their mean free path is supposed to be many times the distance between atomic centers. The variations of path are then computed in terms of the variations of amplitude, and thus the variations of resistance are obtained and *checked with experimental results.* It is shown that the theory in this form explains Ohm's law, gives the correct temperature coefficient and the most important part of the pressure coefficient, avoids the difficulty of the classical theory with reference to specific heats, indicates a vanishing resistance at low temperatures, leaving open the possibility of super-conductivity, and retains the classical expression for the Wiedemann-Franz ratio. Besides these quantitative checks, the theory is shown to be entirely consistent qualitatively with all the new data; in fact, many of these new results, particularly the effect of pressure and temperature on the relative resistance of solid and liquid, seem to demand uniquely this conception of metallic conduction.

IN a forthcoming number of the Proceedings of the American Academy I shall give the results of measurements of the effect of pressure and temperature on the resistance of 18 elements and several alloys. This is additional to my previous results for 22 elements. The numerical data will be briefly summarized in the Proceedings of the National Academy. In a previous paper in this journal[1] I drew certain inferences

[1] P. W. Bridgman, PHYS. REV., 9, 269–289, 1917.

with regard to the mechanism of electrical conduction from the previous data for 22 substances. In this paper I desire to extend this work in the light of the new data now available.

It will pay to briefly indicate the extent of the new data; the numerical results will be referred to in the following as the need arises. Previous to these measurements there were data for the effect of pressure on the resistance of no metal in both the liquid and the solid state, and the data were known for only one liquid metal, mercury. I have now obtained results for the variation of resistance under both pressure and temperature of the following six metals in both the liquid and the solid states: mercury, lithium, sodium, potassium, gallium, and bismuth. Previously none of the alkali or alkali earth metals had been measured; it is just these for which the largest pressure effects would be expected. I now have the data for lithium, sodium, potassium, calcium, and strontium. The effect of pressure was not known on any of the rare earth metals. I now have the data for La and Nd. The complete list of elements covered by the new work is: Li, Na, K, Mg, Ca, Sr, Hg, Ga, Ti, Zr, Bi, As, W, La, Nd, C, I, and P (black). Perhaps the most striking of the new results are as follows. Three more metals have been added to the list of those whose resistance increases with increasing pressure; these are Li, Ca, and Sr. Bi and Sb were the only ones previously known. Of these Li was a particular surprise, because its compressibility is so high. The resistance of liquid as well as solid Li increases under pressure. The resistance of liquid Bi, on the other hand, decreases under pressure, although that of the solid increases. I expected that the resistance of gallium would also increase under pressure, because this substance is abnormal in expanding on freezing, but its resistance decreases normally with pressure in both the solid and the liquid states. The effects of pressure on the resistance of Na and K are much larger than for any other metals, Na decreasing 40 per cent. in resistance under 12,000 kg., and K decreasing by 70 per cent. The decreases of these metals are insignificant compared with that of black phosphorus, however, which decreases to 3 per cent. of its initial resistance under the same pressure. The change is entirely reversible. The pressure coefficient of carbon has opposite signs in the amorphous and graphitic phases.

In my previous theoretical paper I directed especial emphasis to one fact brought out by the previous measurements, namely that the most important single factor in affecting the resistance of a metal is without doubt the amplitude of vibrations of the atoms. It turned out that the relative change of resistance under a change of either temperature or pressure was equal approximately to twice the corresponding change

of amplitude of atomic vibration. The relation was fairly accurate for all metals with regard to the temperature coefficient (which is nearly the same for all metals and equal to the reciprocal of the absolute temperature), and was not so exact for changes of pressure, but nevertheless was somewhat more accurate than any other relation previously proposed for the pressure change. This fact seemed to me to indicate that the mechanism of conduction was by a passage of electrons from atom to atom through the substance of the atom itself. The atom is to be thought of as normally offering no resistance to the passage of the electron (super-conductivity at absolute zero), but resistance may be encountered in passing from atom to atom. The assumption of paths within the atom which are resistanceless need occasion no alarm in these days of non-radiating quantum orbits. In fact there may be an intimate connection between the two. I had called this theory the "gap" theory of resistance. Not only the quantitative fact that the resistance varies as the square of the atomic amplitude, but a large number of qualitative facts also, were in accord with this point of view. These qualitative facts were many of them brought into line by the conception that the "gap" may function in two ways. At large mean distances of separation of the atoms it may happen that passage of electrons from atom to atom is made easier by temperature agitation of unusual violence, which brings the surfaces of the atoms closer together than normal during part of their vibration, whereas at small mean distances of separation, the passage of electrons is on the average hindered by temperature agitation. Except for the deduction of the expressions for the variation of atomic amplitude with temperature and pressure, the theory as hitherto expounded was qualitative rather than quantitative. In particular, I did not attempt to give any detailed picture of the way in which the gaps between atoms might offer resistance to the passage of electrons, or what the character of the resistance might be. The theory as previously given also made no attempt to explain the Wiedemann-Franz ratio, although I pointed out that an explanation was not inconsistent with the elements of the theory.

It is now possible to cast this point of view into quantitative form, at least as far as temperature variations of resistance go; the pressure changes cannot be so easily dealt with for a reason that will appear later.

In the first place it is interesting to observe that the proportionality of resistance to the square of the amplitude of atomic vibration holds also at low temperatures; the previous considerations were entirely confined to ordinary temperatures. It has been noticed by Grüneisen,[1]

[1] E. Grüneisen, Verh. D. Phys. Ges., 15, 186–200, 1913.

after an examination of the best data, that down to very low temperatures the quotient of resistance by absolute temperature is proportional to specific heat, the factor of proportionality being different for different metals. Let us compare this experimental variation of resistance with a variation as the square of the amplitude of atomic vibration at low temperatures. The specific heat formula which best fits the facts at low temperatures is that of Debye. This gives an energy content proportional to the fourth power of the absolute temperature, and of course a specific heat proportional to the third power, since the specific heat is obtained by differentiating the energy. We have, therefore, the expressions,

$$E = At^4 \qquad \text{and} \qquad C = 4At^3,$$

where E and C are the energy and the specific heat of the atom respectively. We may also express the energy of the atom in terms of its frequency and amplitude of vibration. This gives the equation

$$2\pi^2 m\nu^2\alpha^2 = At^4,$$

where α is the amplitude, and ν the frequency of atomic vibration. Now differentiate this expression logarithmically with respect to t, giving

$$\frac{2}{\alpha}\left(\frac{\partial\alpha}{\partial\tau}\right)_p = \frac{4}{\tau} - \frac{2}{\nu}\left(\frac{\partial\nu}{\partial\tau}\right)_p.$$

The value of $1/\nu(\partial\nu/\partial\tau)_p$ was found in the previous paper. Substituting this value gives

$$\frac{2}{\alpha}\left(\frac{\partial\alpha}{\partial\tau}\right)_p = \frac{4}{\tau} - 2\left(\frac{\partial v}{\partial\tau}\right)_p^2 \Big/ C_v\left(\frac{\partial v}{\partial p}\right)_\tau.$$

An examination of the previous deduction of $1/\nu(\partial\nu/\partial\tau)_p$ shows that the connection between temperature and energy did not enter, so that the same expression is valid at low as well as high temperatures. But now it is an empirical fact that at low temperatures the compressibility approaches a constant value, and that the ratio of the thermal expansion to the specific heat also approaches a constant value. Hence in the limit, $(\partial v/\partial\tau)_p^2/C_v(\partial v/\partial p)_\tau$ becomes proportional to $(\partial v/\partial\tau)_p$, and hence vanishes. We have, therefore, at low temperatures,

$$\frac{2}{\alpha}\left(\frac{\partial\alpha}{\partial\tau}\right)_p = \frac{4}{\tau}.$$

At high temperatures $2/\alpha(\partial\alpha/\partial\tau)_p$ was also proportional to $1/t$, but the factor of proportionality was 1 instead of 4.

Now our previous empirical observation was that $2/\alpha(\partial\alpha/\partial\tau)_p$ gives the variation of resistance with temperature. Compare this with

Grüneisen's empirical observation. He has

$$R/t = BC,$$
$$R = BCt$$
$$= 4ABt^4,$$

where R is the resistance and B a constant. Hence

$$\frac{1}{R}\frac{dR}{dt} = 4/t,$$

which checks precisely with the value of twice the variation of atomic amplitude.

Although we have found a variation at high and low temperatures proportional to the variation of the square of the amplitude, we cannot set the resistance at all temperatures equal to a constant times the square of the amplitude (that is if Grüneisen's observation is correct) for the factor of proportionality changes on passing from low to high temperature. The relation of proportionality must therefore fail at some intermediate temperature.

Returning now to the task of making the "gap" theory more definite, we in the first place make specific the action of the gap in imparting resistance by picturing precisely the same sort of mechanism as that operating during a collision of the classical theory. There cannot be any doubt that the electron encounters difficulty in getting free from the atom when the atoms are separated from each other as in a gas, for there is a definite ionizing potential, which involves an amount of energy large in comparison with that ordinarily available in the form of temperature energy of agitation. It is also evident that in some way the electrons do get free from the atoms in the solid state, because the solid is conducting for any E.M.F., no matter how small, and at low temperatures its Ohmic resistance vanishes. This means that under certain conditions forces may act on the electron when in the act of passing from one atom to another, whereas while the electron is passing through the interior of the atom no forces act. During the action of a force between electron and atom, there is a chance for the transfer of energy from one to the other, so that we have a tendency to equipartition. In other words, the "gap" produces resistance by interfering with or terminating the free path of the electron, precisely as did the "collision" in the classical theory. The amount of interference with the free path will depend on the amplitude of vibration, and so the resistance will depend on the amplitude. Such a picture as this enables us to carry over immediately much of the analysis of the classical theory. For instance, the classical ex-

pression for the resistance at once holds, namely,

$$R = \frac{2m}{e^2}\frac{v}{nl}, \qquad (1)$$

where e is the charge on the electron, m the mass of the electron, n the number of electrons describing paths per cm.^{3}, l the mean free path of the electron, and v the undisturbed velocity of translation of the electron.

Ohm's law at once follows on the same basis as in the classical theory. Previously I had to leave Ohm's law unexplained, with only the remark that there was no inconsistency.

This picture of the mechanism of interference with the free path is so different physically from that of the classical theory, and yet gives such similar results when substituted into the mathematical expression, that it will pay to stop for a moment to inquire what is our justification for setting up so definite a picture. In particular, we may compare this picture with that of Wien,[1] who also has a free path mechanism of conduction and Ohm's law, but imagines the electrons interfered with in their flight by collisions with the centers of the atoms. Many facts are equally understandable from either point of view. There are a number, however, which are more naturally explained by supposing that the interference encountered is in passing through the surface from one atom to the next. Many of these facts will be given later; I mention only the two most striking. In the first place it is a universal fact that if a metal changes form, as by melting, or by a polymorphic transition, the phase with the smaller volume has the smaller resistance. This holds for all known normal meltings, in which the liquid has the larger volume, and also holds for the abnormal meltings of Bi, Sb, and Ga, in which the liquid has the smaller volume, and also the smaller resistance. It holds also for the normal polymorphic transitions of Zn and Ni, and for the abnormal transition of Tl, in which the high temperature phase has the smaller volume. Now this is difficult to understand from Wien's point of view, for it would seem that in the phase with the smaller volume there must be more chance of collision with the atomic centers, and so a higher resistance. The difficulty cannot be turned by supposing that the energy of temperature agitation is different in the two phases, for the specific heat of liquid bismuth is greater than that of the solid, and therefore in the liquid the atoms have more chance of terminating the free path both because they are more numerous, and because their amplitude of temperature agitation is greater, so that they effectively cover more territory. On the other hand, the fact that the resistance of the phase

[1] W. Wien, Columbia Lectures, 1913, 29–48.

with the smaller volume is the smaller is most naturally explained from the "gap" point of view simply by the observation that the atoms are closer together, and the electrons find it easier to pass from one to another. Another fact not readily understandable from Wien's point of view is that the temperature coefficient of resistance of the liquid is nearly always less than that of the solid. In some cases it may be very much less, or may even be negative, instead of positive. Now the increased violence of temperature agitation in the liquid, as shown by the higher specific heat, would seem to demand a higher temperature coefficient, according to Wien's view. There is a most natural explanation in terms of the gap theory, as has been explained in the preceding paper, or as will be elaborated further in the following.

So much for the mechanism by which resistance is produced, and the explanation of Ohm's law. To get further, we have to know the precise manner of variation of n, l, and v with temperature and pressure. The classical theory supposed that v was the value given by the equipartition of energy, treating the electron as a gas particle, that l was at least of the order of magnitude of the distance between atomic centers, and n could look out for itself, being determined by the necessities of the case. The weaknesses of the old theory are well known. One of the most serious is that the n needed to give the observed values of specific resistance is of the order of magnitude of the number of atoms itself, or even may be considerably in excess, which leads to the insuperable difficulty of the specific heat. An n of the same order of magnitude was indicated by the application of the theory to the optical theory of metals. Many attempts have been made to avoid this difficulty, but as yet without success. Nearly all attempts at replacing the classical theory have failed in the endeavor to give even an approximate explanation of the Wiedemann-Franz ratio. In order that the thermal conductivity of a metal shall have its high value, and in order that the Wiedemann-Franz ratio may have approximately its experimental value, it seems to be necessary to suppose not only that temperature energy is carried by the electrons, but that the amount so carried is precisely the amount which would be carried if the energy of the electrons were the equipartition energy on the old classical basis.

Apparently the most promising attempt at another explanation of the Wiedemann-Franz ratio is the recent theory of Borelius,[1] which does give a result of approximately the right magnitude. But although his explanation may be mathematically satisfying, I do not think that it can be considered satisfactory physically. His explanation is on the

[1] G. Borelius, Ann. Phys., 57, 278–286, 1918.

basis of the Debye theory of heat conduction.[1] Debye's theory is that heat is conducted by elastic waves, which are dissipated by atomic irregularities. A quantity analogous to the mean free path may be defined in terms of the rate of dissipation, so that the higher the rate of dissipation the shorter the equivalent free path, and the less the conductivity. Debye has applied this theory to insulators, like rock salt, and has been able to account for the experimental fact that the conductivity varies inversely as the absolute temperature. Borelius has taken over this theory for metallic conduction, by means of the assumption that the amount of energy dissipated at each atom as the elastic wave passes over it is equal to the ratio of the thermal energy of the atom to that of the electron. This gives a thermal conductivity of the right order of magnitude. In combination with his theory of electric conduction he gets a Wiedemann-Franz ratio of approximately the right magnitude, and with the right dependence on temperature. But he has neglected the dissipation of the elastic waves due to the atoms. If Debye's analysis is correct, this should be present in the metal as well as in an insulator, and is very much larger than the dissipation supposed by Borelius. Hence, taking account of the neglected atomic dissipation, the thermal conductivity of a metal would turn out to be actually less than that of an insulator, for there is dissipation not only by the atoms but also by the electrons.

Compared with the classical conception of conduction as performed by a swarm of electrons playing in the free spaces between the atoms, the view at which we have arrived of conduction as performed by electrons passing freely through the substance of the atoms places us in a much more advantageous position, for it allows the possibility of very long free paths (in fact at absolute zero there is no resistance to the motion and the paths may be indefinitely long), and hence enables us to get along with many fewer electrons. In this way the specific heat difficulty may be avoided. But in order to account for the facts of thermal conduction, it seems necessary to take over the classical idea that the electrons are moving with the energy of gas particles at the same temperature.

The theory developed here proceeds on the following assumptions. We take over the classical expression (I) for resistance in terms of n, l, and v. We suppose that v is the same as that given by the classical equipartition theory, and we suppose l large enough to avoid the specific heat difficulty by allowing a small value of n. We try to deduce from our fundamental expression the variation of resistance with temperature

[1] P. Debye, Wolfskehlstiftung Vorträge, B. G. Teubner, 1914, 17–60.

and pressure in terms of the variations of v, which are given by the classical expression, and the variations of l, which we get from our picture of the electrons jumping from atom to atom across a region capable of exercising interference with the path. The quantity n we suppose to stay constant. One reason for this supposition is that we do not need to assume any variation in order to account for the facts. However, it does not seem unplausible that n should stay approximately constant. As temperature is increased, the tendency to an increased n due to the increased chance of getting an electron out of an atom by increasing violence of collision is counterbalanced by the increasing difficulty of getting the electron out of the atom because of the increased distance of separation of atomic centers. Or again, it may be that the number of migrating electrons is determined by a sort of spontaneous atomic disintegration in the outer part of the atomic structure, over which changes of temperature or of pressure can have no control.

There is evidence as to the magnitude of n given by optical theories of metals. We shall for the present merely disregard this. We are much more justified in doing this now than we would have been several years ago, for quantum theory has made it exceedingly uncertain whether we are justified in keeping our old model of a vibrating electron as a source of light. Quantum theory has shown,[1] for example, that the success of the electron in giving the classical expression for the Zeeman separation was due to a quite accidental cancelling out of the factor h from the result, and that the factor h will not so conveniently cancel out in treating certain other phenomena, as the Stark effect. Furthermore, the number of electrons demanded by optical theory must now seem impossibly high in the light of our knowledge of the structure of the atom, so that we are probably justified in disregarding the line of attack from the optical side until quantum theory has become more developed.

Concerning our assumption of the equipartition value for v, putting it equal to $\sqrt{2(\kappa\tau/m)}$, there cannot be much question at moderate temperatures, so long as n is small enough to avoid the specific heat difficulty, but there may be question as to what the limit of temperature is to be. It is possible that the classical expression for the energy may hold to lower temperatures for the electron than for the atom. Hydrogen or helium at low temperatures in the gaseous condition in contact with the solid walls of the container continues to have the classical energy, although this is no longer the equipartition energy as compared with the atoms of the solid walls. However, the classical expression cannot continue to hold indefinitely to extremely low temperatures, for even a few

[1] A. Sommerfeld, Atombau und Spektrallinien, Vieweg, 1919, 422–440.

electrons with the classical energy would ultimately make trouble with the specific heat. We will not concern ourselves further with the situation at extremely low temperatures except to remark that the assumptions of this theory will have to be modified, and to indicate that such a modification is not at all impossible. For instance, the electron in coming from the atom, leaves behind it a positively charged ion. As the electron wanders through the metal it must at some time come again in contact with an ion, and may recombine with it. Throughout the metal, therefore, there is going on a continuous process of emission of electrons by the atoms and reabsorption. Now the mechanism of the ejection of the electron by the atom within the solid is something which we do not understand, but it is not unlikely that it is determined from within the atom, and is not greatly affected by outside conditions, at least as far as the velocity of emission is concerned. The electron may, therefore, be expelled with a definite velocity independent of temperature. It then travels from atom to atom, and is jostled about, until it ultimately acquires the equipartition energy. The time required to accomplish equipartition may well be less the greater the violence of agitation. Presently, the electron recombines with an ion. At low temperatures, because of the feebleness of temperature agitation, it may be that the electron is absorbed before it has acquired the equipartition energy, or that it has possessed this energy for only a relatively short time. The specific heat difficulty will not then appear, and the conductivity will be even higher than that corresponding to Grüneisen's empirical observation, because the mean velocity will not decrease so rapidly with decreasing temperature. And it is of course a fact that some metals show superconductivity at extremely low temperatures, and that superconductivity is not covered by Grüneisen's formula.

So much for the assumptions with regard to v and n. Now for a deduction of the variation of l, the mean free path.

There is a point implied in our assumption of a long free path which may detain us for a moment. We have pictured the electron as passing through the substance of many atoms before its path is arrested. Now in a cubic crystal this means that the electronic paths may be confined to three directions mutually at right angles, along the crystalline axes. The classical deduction for the expression for conductivity assumes of course that the directions of the paths are distributed at random throughout space. Is a specialized distribution in only three directions consistent with the known fact that the resistance of a cubic crystal is independent of the direction? A simple analysis immediately shows that there is no inconsistency here. Imagine the applied electric force making

the angles α, β, and γ with the crystal axes, and find the contribution made to the conductivity by the electrons moving along the X axis. The component of force along this axis is $E \cos \alpha$, so that the velocity imparted to the electron is smaller by the factor $\cos \alpha$ than it would be if the force were along the axis. And the component of this velocity in the direction of the force is further diminished by the factor $\cos \alpha$, so that the total contribution of the electrons moving along the X axis is diminished by the factor $\cos^2 \alpha$. Similarly the electrons moving along the other two axes contribute to the conductivity terms equal to $\cos^2 \beta$ and $\cos^2 \gamma$. The sum of the three terms is 1, which is thus independent of direction.

On the other hand, there is no necessity in supposing that a single free path is straight; the electrons may perform Virginia reels about the nuclei of the atoms in quantum orbits. Something of this sort probably has of necessity to take place in non-cubic crystals.

We return to the question of the variation of l. For the moment we consider that the solid is maintained at constant volume. As temperature is increased, the amplitude of vibration is increased, and it may be that during part of the vibration the atoms are so far separated that the electron cannot pass. That is, when the atoms are in vibration, there is a certain chance that the free path of the electron may be terminated in the passage from one atom to the next. This chance is a function both of the amplitude of atomic vibration, and of the mean distance of separation of atomic centers. Call β the chance that the path will be terminated, r the mean distance of separation of atomic centers and α the amplitude of atomic vibration. Then we may expand the unknown function in powers of α and write

$$\beta = f(r, 0) + \alpha \frac{\partial f(r, 0)}{\partial \alpha},$$

provided that α is small. α may be calculated, and it turns out that for ordinary temperatures α is a small fraction of r. Now it is our fundamental assumption that when the atoms are at rest no resistance is encountered in passing from atom to atom, provided that the distance of separation of atomic centers is not too great, as we suppose it is not for a solid. This means that $f(r, 0) = 0$, and we may write

$$\beta = \alpha \frac{\partial f(r, 0)}{\partial \alpha}.$$

That is, so long as the volume is kept constant, the chance that the path will be terminated in passing from atom to atom is proportional to α. In unit distance the chance that the path will be terminated is equal to β

multiplied by the number of atoms in unit length, again giving a constant times α. Now the probability definition of the mean free path is merely the reciprocal of the chance that the path will be terminated in unit distance, so that we have at once

$$l = \text{Const}/\alpha.$$

To compute the variation of l at constant volume it is sufficient to calculate the variation of α at constant volume.

Now the variation of α was already computed in the previous paper. The expressions there given were for the variation of α with pressure and temperature, but of course from these derivatives the derivative at constant volume may at once be found if we know the compressibility and thermal expansion of the substance.

The computation of the change of amplitude is the only place in the theory into which the quantum hypothesis explicitly enters. The assumptions at the basis of the computation were these. The energy of the atom is the classical amount, $\kappa\tau$ (so that the deduction does not hold for low temperatures), the frequency is a function of volume only (the forces on the atoms are on the average a function of volume only), and the entropy of the atom is the same as the entropy of an ideal linear oscillator at the same temperature, which by quantum considerations is shown to be a function of τ/ν only.

For convenience of reference the values previously found are reproduced here.

$$\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial p}\right)_\tau = -\left(\frac{\partial v}{\partial \tau}\right)_p \Big/ C_v,$$

$$\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial \tau}\right)_p = \frac{1}{2\tau} - \left(\frac{\partial v}{\partial \tau}\right)_p^2 \Big/ C_v\left(\frac{\partial v}{\partial p}\right)_\tau.$$

The change of amplitude at constant volume may at once be found

$$\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial \tau}\right)_v = \frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial \tau}\right)_p + \frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial p}\right)_\tau\left(\frac{\partial p}{\partial \tau}\right)_v.$$

But

$$\left(\frac{\partial p}{\partial \tau}\right)_v = -\left(\frac{\partial v}{\partial \tau}\right)_p \Big/ \left(\frac{\partial v}{\partial p}\right)_\tau.$$

Substituting this value above gives at once the simple result

$$\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial \tau}\right)_v = \frac{1}{2\tau}.$$

Now let us find the change of resistance at constant volume with

temperature. Our formula for the specific resistance is

$$R = \frac{2m^{1/2}(2\kappa)^{1/2}}{e^2 n}\,\frac{\tau^{1/2}}{l}$$

$$= \text{Const}\,\frac{\tau^{1/2}}{l}.$$

Differentiating this logarithmically gives

$$\frac{1}{R}\left(\frac{\partial R}{\partial \tau}\right)_v = \frac{1}{2\tau} - \frac{1}{l}\left(\frac{\partial l}{\partial \tau}\right)_v.$$

Using the value found for l in terms of α gives

$$\frac{1}{l}\left(\frac{\partial l}{\partial \tau}\right) = -\frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial \tau}\right)_v,$$

and substituting this gives

$$\frac{1}{R}\left(\frac{\partial R}{\partial \tau}\right)_v = \frac{1}{\tau}.$$

That is, the temperature coefficient of resistance at constant volume is the reciprocal of the absolute temperature. This of course, is in accord with the experimental facts. In my preceeding theoretical paper I gave the values of $1/R(\partial R/\partial \tau)_v$. The difference between $1/R(\partial R/\partial \tau)_v$ and $1/R(\partial R/\partial \tau)_p$ is not large, but is in the direction to make $1/R(\partial R/\partial \tau)_v$ even more nearly equal to $1/\tau$ than $1/R(\partial R/\partial \tau)_p$.

The change from the point of view of the last paper is to be noticed. We previously thought of the change of α as the only significant feature, and noticed that $1/R(\partial R/\partial \tau)_v$ was equal to $2/\alpha(\partial \alpha/\partial \tau)_v$. We did not see any particular reason why the change of resistance should be proportional to twice the change of α, that is, why the resistance should be proportional to the square of the amplitude, although we advanced reasons which made it seem not improbable. Our present expression for R demands that we analyze the change of resistance into two effects; one is a temperature effect, due to the term $\tau^{1/2}$, and the other is an effect due to the change of l, which involves the change of α. Now it turns out that the change of l with α, which again changes with temperature, is of such a nature that the sum of the two effects is as before exactly equal to twice the change of α. Hence the previous analytical result stands, but our physical analysis is different.

We now let fall the condition that the volume be kept constant during the change, and consider the change of resistance with temperature at constant pressure, which is the coefficient usually directly determined by experiment. The formal work of differentiation of our fundamental

expression for resistance may be carried through as before. We encounter difficulty in determining the variation of the mean free path with temperature, for the condition no longer holds that the volume is constant, and β, the probability of the termination of the path, can no longer be put equal to a constant times the amplitude, but the factor multiplying the amplitude is an unknown function of the volume. I have not found any universal assumption as to the dependence of β on the distance of separation of atomic centers which seems to be plausible, but it would seem that β might vary in any way with the structure of the particular atom. For instance, it is conceivable that as the distance of separation of atomic centers is increased the atoms acquire rotational movement with respect to each other, so that parts of the atomic surfaces are brought into opposition which are not so favorable for the passage of electrons. The magnitude of this effect will depend entirely on the structure of the particular atom. Or again, as the atoms are brought closer to each other, the ease of passage will be affected by the deformation of the atom, and this again varies in an unknown way from element to element. The only fact about this unknown effect of changing distance between atomic centers which it seems fairly safe to assume as common to most elements is the sign; it is likely that as the atoms are brought closer together at constant amplitude the ease of passage is increased, and so the mean free path is increased.

The unknown effect of changing volume will evidently be much greater for those changes of volume relatively large compared with the change of amplitude. This means that the unknown effect will be much larger for changes of pressure than for changes of temperature. The magnitude of these changes was considered in the previous paper. It was there shown that the pressure coefficient of amplitude is from six to nine times the pressure coefficient of distance of atomic separation (linear compressibility), whereas the temperature coefficient of amplitude was from 50 to 650 times as great as the temperature coefficient of atomic separation (linear dilatation). In view of the largeness of this latter ratio, it would seem that we are justified for most substances in neglecting the effect as far as variations of temperature at constant pressure are concerned. We will, as a matter of fact, make this assumption. We then have

$$\frac{1}{l}\left(\frac{\partial l}{\partial \tau}\right)_p = -\frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial \tau}\right)_p,$$

and analysis exactly like the preceding would give

$$\frac{1}{R}\left(\frac{\partial R}{\partial \tau}\right)_p = \frac{1}{\tau} - \left(\frac{\partial v}{\partial \tau}\right)_p^2 \Big/ C_v\left(\frac{\partial v}{\partial p}\right)_\tau.$$

Now for all solid metals the second term is small compared with the first, and we may neglect it. We may now replace R, the specific resistance by ρ, the observed resistance (that is, the resistance of a piece of wire with fixed terminals), since the difference of the two coefficients is equal to the linear expansion, which may be neglected in comparison with $1/\tau$, and obtain as our final approximate result

$$\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \tau}\right)_p = \frac{1}{\tau}.$$

This equation, of course, agrees with experimental fact. The usual temperature coefficients are always somewhat larger than $1/\tau$, which is the direction of variation which the above considerations would lead us to expect. I know of no theory which gives the departure of the temperature coefficient from $1/\tau$. Wien's theory is the only one which makes the attempt, and this cannot be regarded as successful. Of course the classical theory does not account for the temperature coefficient at all, but is driven to unplausible assumptions as to the variation of the free path with temperature in order to be consistent.

Now to find the pressure coefficient, we differentiate our expression for resistance with respect to pressure at constant temperature, getting

$$\frac{1}{R}\left(\frac{\partial R}{\partial p}\right)_\tau = -\frac{1}{l}\left(\frac{\partial l}{\partial p}\right)_\tau.$$

The only statement which we can make about $1/l(\partial l/\partial p)_\tau$ is that it is at least as large as $1/\alpha(\partial \alpha/\partial p)_\tau$, so that we expect the pressure coefficient of resistance to be at least as large numerically as $(\partial v/\partial \tau)_p/C_v$ (substituting the value found above for $1/\alpha(\partial \alpha/\partial p)_\tau$) and to be negative. In the previous paper it was shown that the pressure coefficient is given on an average for a large number of metals by $2/\alpha(\partial \alpha/\partial p)_\tau$. Our present theory leads us to replace this by the statement that there is a lower limit one half the value previously found. Now it is true that twice the amplitude represented a somewhat better approximation to the average of all the results than did the expression of Grüneisen, for example, which was the best of other theoretical expressions for the pressure coefficient of resistance, but there were deviations from it in both directions. It is on the other hand true that there are no cases known, either among the substances of the previous work or among the new elements of my more recent work, except those abnormal metals for which the coefficient is positive, and which will be dealt with separately later,* in which the coefficient falls as low as $1/\alpha(\partial \alpha/\partial p)_\tau$. So that although our present theory is not complete because of the unknown element of atomic struc-

ture, it is at least more satisfactory than the former attempt in that it is true without exception.

So much for the formal expression of the theory with regard to temperature and pressure changes of resistance. The classical expression for thermal conductivity is to be taken over without change, and need not be written down explicitly. It gives the right order of magnitude for the Wiedemann-Franz ratio, but is not a complete expression of the facts because it neglects the part of conduction done by the atoms, and hence will be expected to fail particularly at low temperatures, where the atomic conduction becomes important.

The theory as outlined is not inconsistent with a retention of the classical expression for the Peltier heat. In particular the theory is in accord with the classical theory in at least two aspects of thermoelectric phenomena. The magnitude of the pressure coefficient of the thermo-electric force would mean, according to the classical theory, that the number of free electrons does not change much with increasing pressure. This is in exact accord with the assumptions of our theory. Furthermore, the small Peltier heat between a solid and a liquid metal makes it likely that the number of free electrons is not greatly different in solid and liquid. As will be seen in the following, our theory gives an account of the difference of resistance between solid and liquid in terms of a difference of free path only, without supposing a difference in the number of electrons. There was here a weak point of the classical theory, because to account for the great difference of conductivity between solid and liquid it had to suppose a number of electrons much greater in the solid than in the liquid, which was not consistent with the thermo-electric phenomena. At the same time I do not believe that it is desirable to take over entirely the whole classical picture of the mechanism of thermo-electric action. There are, of course ions as well as wandering electrons present in the metal, and these ions may play a part in thermo-electric action, although because of their relatively small translational velocity their part in conduction may be altogether insignificant. That some such modification is necessary is indicated by the fact that the classical expression for the Thomson heat demands that the number of free electrons increase as the square root of the absolute temperature, whereas we have assumed that the number is constant. However, the Thomson heat is relatively a small matter, and I have not attempted to bring it within the range of the present theory. If the above considerations are sound it ought to be possible to get it in without essentially modifying the mechanism which we have set up to account for conductivity.

This is as far as I have got at present with the quantitative develop-

ment of the theory. Its advantages are obvious. It gives an unforced and inevitable description of the variation of resistance with temperature, predicts the right sign and an important part of the numerical magnitude of the pressure coefficient, avoids the difficulty of the classical theory with the specific heats while retaining the classical explanation of the Wiedemann-Franz ratio, and leaves open the possibility of much greater conductivity or even superconductivity at low temperatures, which was not possible to the classical theory. Further development of this theory seems to demand more intimate knowledge of atomic structure than we have at present.

But in addition to these quantitative facts, the theory is able to bring a very large number of facts qualitatively into line, as was emphasized in the preceding paper. In the following I give a survey of the new facts brought out by the new experimental work, and the relation of the theory to these facts.

1. The view of conduction as due to the passage of electrons from atom to atom through the substance of the atom receives confirmation from a group of phenomena not considered in detail in the previous paper, namely the phenomena of the resistance of alloys. Alloys fall into two main groups, according as they do or do not form mixed crystals. Those alloys which do not form mixed crystals solidify by the separation of the two components each in a pure condition, so that the solid alloy consists of a mechanical mixture of microscopic crystals of the two components. It would be expected that the resistance of a mixture of this sort would be the mean of the resistance of its components, and it is indeed the experimental fact that the resistance of such alloys can be computed by the rule of mixtures. The other class of alloys is one in which mixed crystals are formed, the atoms of the two metals entering side by side into the same crystal edifice. This is possible because of a certain degree of resemblance of the two kinds of atoms. The resemblance is not complete, however, and the indiscriminate use of either kind of atom in the crystal edifice is possible only with a certain amount of distortion in the final result. It is a fact that most mixed crystals will not accept an unlimited amount of the foreign ingredient, but the two atoms will crystallize side by side only up to certain limiting proportions. In the mixed crystal structure we would expect, therefore, a certain amount of imperfect fitting between adjacent atoms, with the result that the electrons encounter difficulty in passing from atom to atom, so that, because of the extra resistance of the "gaps" between the atoms, the resistance of the alloy is greater than that computed from the components. This is in exact accord with the experimental facts.

Furthermore, the temperature coefficient of alloys is strikingly in accord with this view. The temperature coefficient of those alloys which do not form mixed crystals is the same as that of the pure metals, which is what we would expect, but the temperature coefficient of those alloys in which there are mixed crystals is very much less than that normal to a pure metal. A large number of binary alloys satisfy Matthiesen's rule in this regard. If we write the resistance of the alloy in the form

$$R_A = R_M + \Delta R,$$

where R_A is the resistance of the alloy, R_M the resistance which would be computed according to the rule of mixtures, and ΔR the resistance which must be added to the computed resistance to give the actual resistance, then Matthiesen's rule states that the temperature coefficient of the alloy is given by the expression

$$\frac{dR_A}{dt} = \frac{dR_M}{dt}.$$

This means that $d\Delta R/dt = 0$, or the additional resistance is not affected by changes of temperature. This is in precise accord with our point of view, for the additional resistance (ΔR) is due to the lack of perfect fit between adjacent atoms of different kinds, and this would be expected to be relatively little affected by changes of temperature.

It is also an experimental fact that the resistance of alloys tends to a finite value at 0° Abs., instead of vanishing like that of a pure metal. This merely means that the gaps between the different kinds of atoms persist to low temperatures, as we would certainly expect.

2. The fact that the resistance of a metal increases on hard drawing is consistent with this point of view. During hard drawing the crystal grains are broken up, and the fitting of the atoms is rendered less exact on the average. This is proved by the universal fact that during hard drawing the density of the metal decreases. But such a disarrangement of the fitting of the atoms means an increased chance of interference when the electrons pass from atom to atom, and consequently an increase of resistance.

Tammann[1] has given an explanation of this fact on the basis of a difference of resistance in different directions within the crystal grains. It would seem, however, that this explanation must be rejected, because the resistance of a cubic crystal is independent of direction (see Voigt, Krystal Physik, pp. 311, 313), whereas the phenomena of increase of resistance on hard drawing is shown by all metals, regardless of their crystal system.

[1] G. Tammann, Lehrbuch der Metallographie, Leopold Voss, 1914, 117–125.

3. It has been observed that as the proportions of the components change through a series of alloys forming mixed crystals the electrical resistance increases in that direction in which the mechanical hardness also increases. Now an increased mechanical hardness means an increased staggering in the positions of the atoms in the crystalline grains, so that it is more difficult to produce sliding of one part of the crystal on another, and by the same token an increase in the difficulty of the electrons in making the leap from atom to atom.

4. In the previous paper a suggestion was made as to the possible explanation of the positive pressure coefficient of bismuth and antomony. The idea was that the amplitude might increase with increasing pressure instead of decreasing as normal. It was shown in that paper that the fact that both metals expand on freezing indicates that there is a certain relative position of the atoms in which the repulsive forces are unusually large at an unusual distance of separation, and that the atoms crystallize in this relative position. From a grossly material point of view this may be expressed by saying that the atoms have knobs, and that the metal crystallizes with the knobs in contact. On the average, except for these knobs, the bismuth atom may be much like that of other metals. The fact that the repulsive forces are unusually large at an unusual distance of separation of the atomic centers is compensated for by an unusually slow decrease of the repulsive force as the distance between atomic centers is decreased. This slow change of the repulsive force was shown in the preceding paper to be consistent with a decreasing frequency of atomic vibration as the centers are brought closer together, and hence consistent with an increasing amplitude with increasing pressure.

This view receives interesting numerical confirmation from recent work of Grüneisen[1] on the equation of state of solids. He has shown that it is possible to a good degree of approximation to explain the behavior of solids by supposing that at least over a small range the forces between atoms are represented to a sufficient approximation by the expression

$$f = \frac{A}{r^2} + \frac{B}{r^m},$$

where the first represents a force of attraction, and the second a force of repulsion. In general m will be much larger than 2. Its magnitude will give an idea of how rapidly the repulsive force increases as the atoms approach. Grüneisen gives an equation for m in terms of atomic volume, atomic heat, thermal expansion, and compressibility. It is not necessary

[1] E. Grüneisen, Ann. Phys., 39, 257–306, 1912.

to reproduce the expression here. Grüneisen finds that m varies for normal metals from 7.5 for iron to 15.6 for gold. If we apply Grüneisen's formula to bismuth and antimony we find that m for the latter is 3.8, and for the former 4.5, both very much less than for normal metals. Our point of view is confirmed, therefore, that the repulsive forces in bismuth and antimony increase more slowly than normal as the distance between atomic centers is decreased, and therefore we have the possibility of an increasing amplitude with increasing pressure and so an increasing resistance.

5. The fact that the temperature coefficients of solid bismuth and antimony are normal is quite in accord with our view, for with rising temperature we have seen that the effect of temperature on increasing amplitude quite overshadows any pure volume effect, so that we would expect the temperature coefficients of all metals to be nearly the same, irrespective of the behavior of the pressure coefficient.

6. The fact that the pressure coefficient of liquid bismuth is normal in being negative is significant. It indicates that the positive coefficient of the solid is in some way connected with the crystalline structure. The picture which we have given of the mechanism of the positive coefficient of the solid has this property, for we have ascribed the increasing amplitude with increasing pressure in the solid to the fact that the atoms are held in fixed orientations with respect to each other, and that in this particular orientation the forces are abnormal in character. Such a fixity of orientation is possible only in the crystal. In the liquid there is no definite relation of orientation, the localities of abnormal force play a relatively unimportant part, and the liquid behaves normally. Not only is the pressure coefficient of liquid bismuth normal, but the temperature coefficient is also normal for a liquid, and is less than that of the solid, and less than the reciprocal of the absolute temperature.

7. It was shown in the previous paper that the abnormality of force between the atoms of bismuth made possible a crystal with abnormally large volume, and hence a crystal which expands on freezing. By analogy I was prepared for a positive pressure coefficient in gallium, which is also abnormal in expanding on freezing. The facts are the opposite, however, and the pressure coefficient of gallium is negative, as is normal. Our previous argument was entirely qualitative, however, and merely indicated the tendency toward an increasing amplitude with increasing pressure without setting up a criterion as to whether the tendency might be strong enough to counteract the normal tendency in the opposite direction or not. In the absence of a definite criterion therefore, we have only the right to expect the same tendency in gallium without

actually being sure whether the tendency will be strong enough to make the coefficient positive. Now as a matter of fact, if the expected magnitude of the pressure coefficient is calculated by the method of the previous paper, it will be found that the discrepancy for gallium is greater than for any other substance with negative coefficient, the calculated value being about twice the observed value. This means that the pressure coefficient is much less than we would expect from the behavior of normal substances, so that here we evidently have the tendency which we are looking for toward a positive coefficient.

In other respects the behavior of gallium is as we would expect. The temperature coefficient of the solid is normal, and the temperature coefficient of the liquid is also normal in being less than that of the solid. Furthermore, the pressure coefficient of the liquid is normal, and is greater numerically than that of the solid, as we would expect, because the tendency to abnormality in the solid is due to a particular orientation of the atoms, and this disappears in the liquid.

8. The behavior of lithium is of a type not shown by any other substance yet known. Its pressure coefficient is positive in both solid and liquid, and the coefficient of the liquid is greater than that of the solid. The temperature coefficient behaves normally. Furthermore, the melting of lithium is normal, in that the liquid has a larger volume than the solid. The data seem not to have been previously determined. In the paper on new resistance data will be found the melting data for lithium which I determined for this particular purpose. I have followed the melting curve up to 8,000 kg., and it seems normal in every respect. Our picture of the mechanism of conduction in lithium must probably, therefore, be different from that of bismuth or antimony or gallium. It is of course possible that the explanation of the abnormal coefficient follows on the same lines as for other metals, namely an increasing amplitude of atomic vibration with increasing pressure. If this explanation is adopted, the abnormality of the atom must not be thought of as confined to certain localities which function only in the crystalline phase, but the abnormality must be one of the atom as a whole, for the liquid as well as the solid is abnormal. This is not an impossible view, for if m be calculated for lithium by Grüneisen's formula, the value 5 will be found, which is low compared with most other metals, but is not low compared with bismuth and antimony. But now the question arises as to the interpretation to be put on the value of m. m itself merely is a measure of the rapidity with which the repulsive force increases as the atom is approached. A low value may be due either to an abnormality of a particular part of the atom, as we have supposed the case with bismuth

and antimony, or it may indicate that the atoms are separated by more than the usual distance, and the repulsive force is not important. We would expect the latter view to be more nearly correct for substances which are very compressible. This seems indicated by the fact that m is as small for the very compressible elements sodium and potassium as it is for lithium, and sodium and potassium have very large negative pressure coefficients of resistance. Now lithium is also one of the most compressible metals. It seems likely to me that the low value of m is more probably connected in some way with the high compressibility than with an abnormality which might result in an increasing amplitude of atomic vibration with increasing pressure.

Now there is a picture of conduction offered by the theories of Wien[3] and Lindemann[1] which gives an alternative explanation, which may quite probably be correct for this particular substance. Their picture is of electrons as well as atoms located on a space framework, and of the space framework of electrons moving bodily through the framework of the atoms when an external force is applied. The mean free path of the electrons in this motion is interrupted by collisions with the nuclei of the atoms. This picture means a positive pressure coefficient of resistance for most substances, for as pressure is increased the channels of passage of the electrons become more restricted. The reason for this is that the closing in of the channels because of the decreasing distance between atomic centers more than neutralizes the opening of the channels because of decreasing amplitude of atomic vibration. This is in spite of the fact that the relative change in atomic amplitude is much greater than the relative change in the distance between atomic centers, because the distance between atomic centers is absolutely much greater than the amplitude. It will be found as a matter of fact that if the relative magnitudes of atomic distance and amplitude and of the changes of atomic amplitude and distance with pressure be calculated for lithium, Wien and Lindemann's picture would lead us to expect a positive pressure coefficient. Evidently the positive pressure coefficient suggested by this picture remains true for the liquid state so long as the electrons continue to move between the atoms.

Apart from the argument from the pressure coefficient, this picture of Wien and Lindemann seems to have a particular probability of being correct for lithium because of its atomic structure. The structure is a simple nucleus consisting of a central positive charge and two electrons rotating about it, and a single other electron connected in some way with the nucleus at a relatively large distance. If the superficial electrons of

[1] F. A. Lindemann, Phil. Mag., 29, 127–140, 1915.

the atom are ever to arrange themselves into a space lattice, it would seem that here is their chance.

The fact that the pressure coefficient of liquid lithium is larger than that of the solid is simply explained as due to the greater compressibility of the liquid. The compressibility of the liquid has never been measured, but there is no case known in which the liquid does have a smaller compressibility than the solid, and furthermore the behavior of the melting curve, its direction of curvature, and the direction of the difference of volume between solid and liquid, makes it almost inevitable that the compressibility of the liquid be greater than that of the solid.

The fact that the pressure coefficient of resistance becomes greater at higher pressures is also intelligible from this point of view. The channels through which the electrons slip are the spaces which are left between atomic centers after the inpenetrable nucleus has been subtracted. Hence a given proportional decrease in the distance between atomic centers will mean a greater proportional decrease in the channel when the atoms are close together (high pressures) than when they are further apart.

9. On the other hand, the increase of the pressure coefficient of bismuth and antimony with increasing pressure is at least perfectly consistent with the picture presented of their conduction as performed by electrons passing through the atoms instead of between them. If reference is made to Fig. 1 of the previous theoretical paper it will be seen that it is quite possible that as the atomic centers approach the restoring force per unit displacement becomes weaker. In fact such a behavior is inevitable at some value of the volume, but we have no criterion for deciding whether this takes place at a volume greater or less than the volume at which the crystal is stable. Now such a weakening of the restoring force when the volume decreases means an amplitude of vibration becoming progressively larger at the higher pressures (smaller volumes), and hence a pressure coefficient increasing with increasing pressure.

10. The pressure coefficients of calcium and strontium were also found to be positive. It is difficult to decide from the data at hand whether this is more probably due to the lithium or the bismuth type of mechanism. The melting data are not known for either of these metals, so that we do not know whether the liquid or the solid has the greater volume. Neither are the data known for the effect of pressure on the resistance of the liquid. It is true that the value of the "m" of the atomic force calculated for calcium is somewhat lower than for the ordinary run of metals, being 6, but the compressibility of calcium is also high, and the m

for strontium is distinctly high, being 9.5. It is, of course, probable that the mechanisms of calcium and strontium are the same, because of their similar positions in the periodic table. I believe that at present the probability is in favor of the lithium rather than the bismuth type of conduction. The melting data would be a great help toward this decision.

11. There is an interesting relation between the temperature coefficients of solid and liquid lithium. At the melting point of lithium, 180°, the temperature coefficient of the solid is 0.00251 (obtained by a linear extrapolation of the resistances at 0° and 100° and therefore somewhat uncertain, but the best that we can do), and the coefficient of the liquid is 0.00150. The ratio of these two coefficients is 1.69, which is almost exactly the ratio of the specific resistance of the liquid to that of the solid at the melting temperature, for which I found the value 1.68.

This can be easily understood in the light of Wien and Lindemann's picture. The resistance of the liquid is higher than that of the solid because the regular channels between the atoms are broken up by the haphazard arrangement of the atoms in the liquid. Now this factor of random arrangement of the atoms in the liquid is one that will persist at all temperatures, producing a permanent difference of resistance between solid and liquid. This can be given mathematical expression. Write the equation

$$R_L = R_S + \Delta R,$$

where R_L is the resistance of the liquid, R_S the resistance of the solid, and ΔR the increment of resistance on passing from solid to liquid. Now differentiate with respect to the temperature

$$\frac{1}{R_L}\frac{dR_L}{dt} = \frac{R_S}{R_L}\cdot\frac{1}{R_S}\frac{dR_S}{dt} + \frac{\Delta R}{R_L}\cdot\frac{1}{\Delta R}\frac{d\Delta R}{dt}.$$

If the experimental relation

$$\frac{1}{R_L}\frac{dR_L}{dt} = \frac{R_S}{R_L}\cdot\frac{1}{R_S}\frac{dR_S}{dt}$$

is to hold, then we must have $d\Delta R/dt = 0$. That is, the increment of resistance when the solid passes to the liquid is independent of temperature, which is what our picture suggested.

12. The same relation between the temperature coefficient of the solid and the liquid holds for some other metals. Thus for sodium at the melting point at atmospheric pressure I found for the coefficient of the solid 0.00414, and for the liquid 0.00310. The ratio of these is 1.34, and I found experimentally the ratio of the resistance of the liquid to that of the solid to be 1.45. Mercury has the largest ratio observed for the

ratio of specific resistance of liquid to solid, and we would therefore expect an unusually small temperature coefficient for the liquid. It is in fact unusually small, being only 0.00090 at 0° C. If we assume that the coefficent of the solid is $1/7$, we find for its coefficient at the melting point (− 38.85°) 0.00409, and for that of the liquid at the same temperature 0.000934. The ratio of these is 4.38 and the ratio of the resistance of the liquid to the solid found by Onnes[1] is 4.22. On the other hand, the relation breaks down for solid and liquid potassium. I find that the temperature coefficient of the liquid is actually slightly greater than that of the solid at the melting point, which is abnormal, whereas the resistance of the liquid is 1.56 times greater than that of the solid. It might be mentioned that my figures for the temperature coefficients of solid and liquid do not agree with those of other observers. The values of Northrup[1] are most favorable to the hypothesis being urged here. He makes the coefficient of liquid potassium less than that of the solid, but the ratio of the two coefficients as found by him is only 1.27, whereas the ratio of specific resistances would demand 1.56.

The only other metals for which I have determined the coefficients, namely bismuth and gallium, cannot possibly satisfy the relation because the temperature coefficients of the solid are greater than those of the liquid, as is normal, but the resistance of the liquid is less than that of the solid. This is because of the abnormal volume relation, so that we are not surprised that the relation breaks down.

There are also some data of Northrup[3] for other metals. For gold his data give a ratio of the temperature coefficients at the melting point of 2.15, and he found for the ratio of the resistances 2.28.

Now with regard to the explanation of this relation between the temperature coefficients, our point of view would lead us to expect an invariable element, unaffected by changes of temperature, in the relative resistances of solid and liquid, and hence, according to the analysis, a ratio of the coefficients equal to the inverse ratio of the resistances. When the solid melts to the liquid the atoms lose their regular mutual arrangement, which becomes haphazard. In passing from atom to atom in the liquid the electrons cannot avail themselves of particularly favorable localities, but must use the average of the entire atom. This element will be a permanent difference between solid and liquid, unaffected by changes of temperature. However, the relations are so complicated that this can be only part of the picture, and in fact the relation

[1] K. Onnes, Kon. Akad. Wet. Proc., 4, 113–115, 1911.
[2] E. F. Northrup, Trans. Amer. Elec. Chem. Soc., 20, 185–204, 1911.
[3] E. F. Northrup, Jour. Fran. Inst., 177, 287–292, 1914.

does not always hold. There is a specific effect due to the "τ" term in our formula for resistance. This may to some extent be neutralized by the tendency of the "gaps" in the liquid condition to function in the second way, decreasing instead of increasing the resistance with increasing amplitude. Furthermore, it is not certain to what extent our deduction of the variation of amplitude with temperature and pressure is valid for the liquid. We assumed that the energy of atomic vibration was either potential or else kinetic energy of to and fro motion. Now in the liquid there is probably some rotational energy as well. The exact relation between the translational and the rotational energies in the liquid is not yet clear, so that we do not know how large an effect to expect on this account. It seems evident, however, that the direction of the effect will be to decrease somewhat the temperature coefficient of amplitude, because part at least of the energy will be rotational, and therefore the translational energy, and so the amplitude, cannot increase as rapidly with increasing temperature as if we had supposed all the energy to be translational.

The entire theoretical significance of this observation as to the ratio of the temperature coefficients of solid and liquid may therefore not yet be completely clear, but at any rate the observation itself is to be kept in mind. I am not aware that this relation has been previously noticed.

13. The considerations of the last section have an application to the temperature coefficient of resistance of the liquid at high temperatures. Write the formula for the resistance in the form

$$R_L = \text{const } \tau^{1/2}/l_L.$$

Now the free path, l_L, is inversely proportional to the chance, β, that the free flight of the electron will be terminated in passing from atom to atom. Let us analyze this chance of termination into two parts. One will depend on the haphazard orientation of the atoms with respect to each other, and will be independent of temperature, and the other will depend on the amplitude, and will, as before, be proportional to the amplitude, if we can neglect the pure volume effect, and the tendency of the gaps to function in the second way. Hence we may write

$$\beta = A_1 + A_2\alpha,$$

which gives

$$R_L = \text{const } \tau^{1/2}(A + \alpha).$$

Differentiate this with respect to the temperature, giving,

$$\frac{1}{R_L}\frac{dR_L}{d\tau} = \frac{1}{2t} + \frac{\alpha}{A + \alpha}\cdot\frac{1}{2\tau}.$$

Hence we see that at high temperatures, where α is large, the temperature coefficient again approaches $1/\tau$, although at lower temperatures it is less than $1/\tau$. These considerations apply only to those liquids for which the functioning of the gaps by decreasing resistance at large distances of separation may be neglected. Now there are liquids for which this last factor is certainly important, as zinc and cadmium, whose temperature coefficients immediately after melting are negative, but there are other liquids, as tin and lead, whose resistance in the liquid shows no peculiarity. The resistance of these liquids increases linearly with the temperature. In fact the linear increase of the resistance of tin is so striking that Northrup and Suydam[1] have proposed this as the basis of a resistance thermometer. Now it is evident that the temperature coefficient of any metal which increases linearly in resistance with temperature must eventually become equal to $1/\tau$, according to the above formula.

14. The very large difference of resistance between solid and liquid mercury is striking, particularly in view of the fact that the change of volume on melting is not larger than usual. This would suggest that the structure of the mercury atom is such that the element of haphazard orientation is particularly important. This element we saw has a tendency to constancy. Consider now the variation of resistance with pressure. I found experimentally that the pressure coefficient of the liquid is less numerically than that of the solid, a somewhat unexpected result. Write

$$R_L = R_S + \Delta R.$$

Differentiate with respect to the pressure

$$\frac{1}{R_L}\left(\frac{\partial R_L}{\partial p}\right)_\tau = \frac{R_S}{R_L}\cdot\frac{1}{R_S}\left(\frac{\partial R_S}{\partial p}\right)_\tau + \frac{\Delta R}{R_L}\cdot\frac{1}{\Delta R}\left(\frac{\partial \Delta R}{\partial p}\right)_\tau.$$

This equation shows at once that the experimental relation

$$\frac{1}{R_L}\left(\frac{\partial R_L}{\partial p}\right)_\tau < \frac{1}{R_S}\left(\frac{\partial R_S}{\partial p}\right)_\tau$$

demands that

$$\frac{1}{\Delta R}\left(\frac{\partial \Delta R}{\partial p}\right)_\tau < \frac{1}{R_S}\left(\frac{\partial R_S}{\partial p}\right)_\tau,$$

which is what our picture led us to expect. Notice that we have disregarded the pure volume effect, so that these considerations would not be expected to apply to a highly compressible metal. Mercury, however, is relatively incompressible, and the considerations have a certain force.

15. A similar consideration applies to the pressure coefficient of alloys. The pressure coefficient of resistance of the alloys of the relatively in-

[1] E. F. Northrup and V. A. Suydam, Jour. Fran. Inst., 175, 153–161, 1913.

compressible metals is usually less than that of the components. This is to be ascribed to a relative insensitiveness to pressure of the feature which is responsible for the increased resistance of the alloy. This feature is the failure of the exact register of the different kinds of atoms, and this feature will be insensitive to pressure unless the atoms themselves are highly deformable. The data for alloys are very restricted, and it is not known how the alloys of such very compressible metals as sodium and potassium would behave under pressure.

16. The same considerations also show why the ratio of resistance of solid to liquid is relatively constant along the melting curve. For the six substances which I measured this ratio suffered relatively little change, although the accuracy of the measurement was not as great as desirable in some cases. This constancy is to be ascribed to the fact that the effect of haphazard orientation in the liquid as opposed to the regular arrangement in the crystal is an intrinsic difference between solid and liquid, and is not affected by temperature and pressure changes.

17. In my previous paper some significance was attached to the fact that the temperature coefficient of liquid mercury at constant volume is negative. This was ascribed to the second manner of functioning of the gaps at the increased volume of the liquid. It was suggested that such might be found to be the case for all liquid metals. At the same time there was no necessity in the suggestion, and there was no criterion which could show whether the second manner of functioning of the gaps would be more important than the normal method for other metals or not. It appears that this is not the case for the new metals. The computation could be made only for sodium, potassium, and bismuth. The values of the thermal expansion are not known for liquid gallium. For lithium, because of the abnormal pressure coefficient of the liquid, the temperature coefficient of the liquid at constant volume is even larger than at constant pressure, and is of course positive. The computation for the three metals above is somewhat uncertain because the compressibility and thermal expansion of the liquids are not known with any great accuracy. A discussion will be found in the new American Academy paper. I find for sodium that the coefficient at constant pressure is 0.00325 against 0.00170 at constant volume; for potassium the respective coefficients are 0.0044 and 0.0025, and for bismuth they are 0.000475 and 0.000015. The uncertainty in the fundamental data is not so great but that there can be no doubt that the coefficients at constant volume of both liquid sodium and potassium are positive, but there may be considerable question in the case of bismuth.

Sodium and potassium are among the most compressible and ex-

pansible of the elements, and it is perhaps not surprising that for them the amplitude continues to function in the normal manner. The example of bismuth makes it not unlikely, however, that the coefficient at constant volume of the ordinary liquid metals may be negative, as it is for mercury, instead of positive. It is in any event significant that the difference between the coefficients at constant volume and constant pressure of all the liquid metals is greater than the difference for the solid, so that the tendency of the gap to function in the second way is manifest.

The second manner of functioning of the gap in liquid mercury may at first seem inconsistent with the relative constancy of the gap under pressure, which we invoked to explain the pressure coefficient of the liquid being less than that of the solid. But it is to be remembered that a change of temperature is always more effective than a change of pressure (compare the magnitudes of the compressibility and the dilatation) so that the two facts are not inconsistent. It is also to be kept in mind that the difference of compressibility between solid and liquid mercury is abnormally small.

18. The behavior of the alkali metals sodium and potassium is of interest because of the unusual magnitude of the effect, and demands special discussion.

In the first place I may mention in the interest of candidness that I had anticipated a possible positive pressure coefficient for potassium on the basis of the similarity of the structure of its atom to that of lithium. According to Langmuir's picture,[1] lithium consists of a positive kernel surrounded by one shell, with a single electron outside that shell, whereas potassium consists of a positive kernel, two surrounding shells, and a single electron outside the two shells. This similarity would seem to give some justification for the expectation of a positive pressure effect. However, the facts are the exact reverse. Not only is the pressure coefficient of potassium negative, as is normal, but it is larger than for any other metal as yet measured. It would seem, therefore, that the greater size of the central portion of the atomic structure of the atom of potassium prevents the arrangement of the superficial electrons in a space lattice, or at least is not favorable to the ready mobility of such a lattice, if it exists. The mechanism of conduction in the case of potassium is probably normal, in that the electrons pass through the substance of the atoms. It must be said, however, that potassium is abnormal in several respects. My measurements make the temperature coefficient of the liquid at the freezing point greater than that of the solid, and the pressure

[1] I. Langmuir, Jour. Amer. Chem. Soc., 41, 868–934, 1919.

effects are abnormal in that the coefficient of the liquid decreases with increasing temperature. It is therefore not impossible that part of the mechanism of conduction of potassium is abnormal.

If the expected magnitude of the pressure coefficients of sodium and potassium is computed on the basis of the preceding paper from twice the pressure coefficient of amplitude, it will be found in both cases that the computed values are very much too small. The observed coefficient of sodium is 1.75 times the computed, and the observed value for potassium is 2.4 times that computed. According to our view, this means an unusually large volume effect in addition to the amplitude effect. Now not only are the compressibilities of sodium and potassium among the highest of the elements, but the change of atomic distance with pressure is an unusually large fraction of the change of amplitude, so that for this additional reason we would expect a large volume effect. If the ratio of the pressure coefficient of amplitude to the coefficient of linear compressibility be computed, by the use of formulas already given, the values 3.8 and 3.6 will be found for sodium and potassium respectively. This is unusually small, we have already seen that the run of values for the ordinary metals is from 6 to 9. Lithium is the only metal which is markedly lower, and for this the ratio is 0.95, but since its conduction mechanism is abnormal, we need not consider it further. It is, then, just for such metals as sodium and potassium that our picture would lead us to expect the largest discrepancy between the amplitude and the total effects, and it is just here that we find them.

Sodium and potassium are also unusual in the large decrease of the pressure coefficient with increasing pressure. Between atmospheric pressure and 12,000 kg. the instantaneous coefficient of sodium has decreased by a factor of 2.5, and that of potassium by 2.6. In the previous work the largest decrease was for lead, with a factor 1.38. The large decreases for sodium and potassium are evidently connected with the unusually high compressibilities. It is a universal result of experiment that those substances with a high compressibility show not only an unusually large absolute decrease of compressibility with rising pressure, but the relative decrease is also unusually high. The actual data have not been determined for sodium and potassium, but we have every reason to expect that the same will be found to hold for them also. This means that the volume effect becomes relatively less at high pressures, and that the pressure coefficient decreases by an unusually large amount.

In addition to the unusually large decrease of pressure coefficient with increasing pressure, there is an unusually large change of the coefficients with temperature. The pressure coefficient increases with rising tempera-

ture by an unusually large amount, whereas the metals of the previous paper showed relatively little change with temperature. The coefficients of sodium at atmospheric pressure increase by a factor of 1.18 between 0° and 80°. The corresponding comparison cannot be made for potassium, because it is not solid over a wide enough range, but a comparison of the results at higher pressures shows that the change for potassium is even more rapid than for sodium. Of course a mathematical consquence of a pressure coefficient changing rapidly with temperature is a temperature coefficient changing rapidly with pressure. This effect again can be explained on the basis of the volume effect. Sodium is unusually expansible at atmospheric pressure, and we would expect a temperature coefficient considerably greater than that computed on the basis of the amplitude effect alone, that is, a coefficient greater than $1/\tau$. And as a matter of fact, the average coefficient of sodium between 0° and 100° is 0.00475, larger than $1/\tau$ by an unusual amount. Now at higher pressures, although the actual measurements have not yet been made, it is exceedingly probable that the volume expansion will show an unusually large decrease, and that sodium will approach more nearly the behavior of the more staid metals. The behavior of the temperature coefficient is in entire accord with this expectation, at 12,000 kg. it has dropped from 0.00475 to 0.00408, a value still somewhat larger than $1/\tau$, but not any larger than for many of the other metals at atmospheric pressure.

These considerations would lead us to expect the temperature coefficient to become not less than $1/\tau$ for any pressure, no matter how high, as long as the atomic vibration continues to function in the normal way in decreasing the probability of an undisturbed flight of the electron. It is conceivable that at exceedingly high pressures the atom itself may become so much compressed that a moderate amount of temperature disturbance is not sufficient to interfere at all with the chance of passage of an electron from atom to atom. The kinetic energy of temperature agitation in such a solid would consist in a quivering of the nucleus within the outer structure of the atom, which is prevented from much superficial motion by the close packing of the adjacent atoms. This state of affairs demands an unusually small thermal expansion. If such a state of affairs should occur, we might expect the temperature coefficient of resistance to become less than $1/\tau$. Now this is actually what does occur with potassium. Its mean temperature coefficient between 25° and 60° decreases from 0.00454 at atmospheric pressure to 0.00184 at 12,000 kg., the final value thus being much less than $1/\tau$. This speculation is most attractive, but more weight cannot be attached to it until

the dilation of potassium has been measured at high pressures. It may be said that if this effect is to exist anywhere, it has the best chance with potassium of the metals measured, since it is by far the most compressible. Potassium is a trifle more than twice as compressible as the nearest metal sodium. And in further support of this view it may be noticed that the temperature coefficient of potassium at 6,000 kg. has dropped only to 0.0034, and it is therefore still a little higher than $1/\tau$ ($\tau = 298°$ Abs.), so that the decrease of the temperature coefficient becomes increasingly rapid at higher pressures, a most unusual behavior for a pressure effect. In caution it may be said that the experimental accuracy in determining the variation of the coefficients of potassium was not so high as for some other substances. It will be most interesting to attempt the verification of this observation on cæsium, which is almost twice as compressible as potassium.

If this observation should be justified by the measurements of thermal expansion, it would be a further most important verification of the view that the free flight of the electrons is interferred with during the passage from one atom to another, and that the interference is not due to the vibration of the nucleus, for the temperature agitation of the nucleus must continue to exceedingly high pressures.

19. The new data cover the non-metallic elements carbon and phosphorus (black). It is probably too early to extend this theory to all non-metallic substances, but it is interesting that in two important particulars the behavior of carbon and black phosphorus is what we would expect. We have thought of the gaps between atoms as functioning in two ways. Normally in metals the electrons leap easily across the gaps, and their passage is made more difficult by temperature agitation. The gaps, when in this condition, have a low specific resistance. But when the atoms are separated by more than the usual amount the electrons usually find difficulty in leaping the gaps, which are thus of high specific resistance, but the passage may be made more easy by increasing violence of temperature agitation, which brings the atoms closer together during part of their vibration. If the gaps are on the average in this second condition, we expect a high specific resistance for the substance as a whole, and a negative temperature coefficient of resistance (at least at constant volume, and presumably at constant pressure). Now these are the facts for both carbon and black phosphorus. Their specific resistance is much higher than that of any metal, but their temperature coefficient is negative.

It is also natural to expect that in such a substance the effect of pressure will be to drive the gaps from the second condition to the first as the

atoms are brought closer together. This means an unusually large decrease of resistance with pressure. This is the fact for black phosphorus. The pressure effect is larger than for any other substance, and is so large that at 12,000 kg. the resistance is only 3 per cent. of its value at atmospheric pressure. Even after this enormous decrease the resistance is still much higher than for metals. The specific resistance of black phosphorus normally is 1 ohm per cm. cube, and therefore at 12,000 kg. it is still 0.03. The temperature coefficient at 12,000 kg. is still negative, but has dropped numerically from 0.0058 to 0.0030, which is in the direction to be expected as the resistance approaches that of a metal in character. The pressure effect on black phosphorus is different in another respect from that of metals, in that the relative coefficient $1/R(\partial R/\partial p)_T$ is very nearly independent of pressure. We saw that for metals this coefficient decreases with increasing pressure, and we gave as the explanation the decreasing compressibility with increasing pressure. It is evident that the mechanism is different in the case of black phosphorus. A constant coefficient would mean that a constant increment of external pressure always drives the same fraction of the total number of gaps from the second into the first group. We would expect this condition to hold as long as the number of electrons in the second group is still large. That this number still is large, even under 12,000 kg., is shown by the fact that under 12,000 kg. the resistance is still very much higher than for metals.

CONCLUSION.

We have now considered a great many of the phenomena of the electrical resistance of metals, and found them all consistent with the view of the nature of electrical conduction previously advanced. The facts considered in this paper were nearly all not known when the view was first suggested and are different in character from any previously available, having largely to do with the pressure and temperature changes of resistance of both the liquid and the solid metal, particularly of the alkali metals, in which the effects are very large.

The view of conduction is that the free paths of the electrons are interfered with in jumping from atom to atom, but that throughout the interior of the atom there is no resistance to their motion. We have put this conception into quantitative form, partly on the basis of the classical theory. At ordinary temperatures we have assumed the equipartition velocity of the electrons, that their number is independent of the temperature, and that the free path varies with the amplitude in a way which may be computed. This gives Ohm's law, the correct temperature coefficient, the Wiedemann-Franz ratio, an important part of the pressure

effect, vanishing resistance at 0° Abs., and avoids the specific heat difficulty. In addition there are certain features which can be discussed only qualitatively; there is a specific effect of changing volume of which we may be pretty sure of the sign, but which depends in an unknown way on atomic structure, and there is a second manner of functioning of the "gap" at large volumes. Within the restricted range open to quantitative discussion the facts are in accord with the theory, and throughout the much wider domain open to qualitative discussion I have not found a single fact which is inconsistent, and many which I believe demand this view uniquely.

The probability seems great that the view contains the most important elements operative in the phenomena of resistance and thermal conduction in metals. This theory does not exclude the presence of other factors which may be important for the thermo-electric and the Hall effects. In fact our theory has made it inevitable that there should be in the metal charged ions as well as the electrons, and there is no reason why these should not be important in certain phenomena, although because of their low velocity of translation they cannot be expected to play a large part in conduction.

THE JEFFERSON PHYSICAL LABORATORY,
HARVARD UNIVERSITY, CAMBRIDGE, MASS.

THE LUMINOSITY OF MERCURY VAPOR DISTILLED FROM THE ARC IN VACUO.

BY NORMAN H. RICKER.

SYNOPSIS.

Afterglow of Mercury Vapor from an Arc in Vacuum.—(1) The *effect of an electric field on the spectrum* was studied. It was found that charged grids decreased the luminosity without changing the color; but when, under certain conditions, the luminosity was increased, there was a change in the relative intensity of the lines. (2) The *velocity of the vapor was measured by a stroboscopic method.* The luminosity was decreased intermittently at a known frequency and the resulting "puffs" as they moved along the tube, were observed stroboscopically. From this velocity and the rate of distillation, the density of the vapor was computed. (3) *Decay of luminosity along the tube* was observed. (4) *Theoretical discussion* of these results leads to the conclusion that recombination of the positive and negative ions produced by the discharge, is the most probable cause of the afterglow.

I. INTRODUCTION.

IT has been observed by a number of experimenters that, when an electrical discharge has been passed through a gas, the luminosity frequently persists for some time after the discharge has ceased. In many cases this luminosity has been shown to be due to a readjustment of chemical equilibrium in the gas, which has been upset by the discharge. This is very noticeably the case in the electrical discharges in oxygen, in hydrogen, and in nitrogen. Strutt[1] has extensively investigated the discharge in these gases and shown that the persistence of luminosity in oxygen is due to the chemical action of ozone, formed in the discharge, with traces of impurities present in the gas. The afterglow in nitrogen was shown to be due to the formation of atomic nitrogen which gives off light upon recombining to form molecular nitrogen, or in reaction with impurities in the gas. Also an active form of hydrogen seems to be formed by the discharge in hydrogen, which active modification, upon interacting with impurities in the gas, produces an afterglow.

However there are many other cases in which a chemical action could hardly be expected to explain the phenomenon. Of this type of afterglow, one of the most interesting examples is the case of mercury vapor.

[1] Proc. Roy. Soc., Vol. 85 (1911), p. 219; Vol. 85 (1911), p. 377; Vol. 85 (1911), p. 537; Vol. 86 (1912), p. 56; Vol. 86 (1912), p. 105; Vol. 86 (1912), p. 262; Vol. 86 (1912), p. 529; Vol. 87 (1912), p. 179; Vol. 86 (1912), p. 531.

It was observed by Stark and Reich[1] in 1903, that, if a side tube were connected to a mercury arc in vacuo, so that the mercury vapor in the arc could, by virtue of its vapor pressure, be blown out into a condensation chamber (Fig. 1), this vapor which distills away remains luminous during its journey away from the arc. If this side tube is not of too small a diameter, the vapor may, under proper conditions, be made to remain noticeably luminous for a considerable distance along its journey. Stark[2] showed that this vapor was intensely ionized, and that the intensity

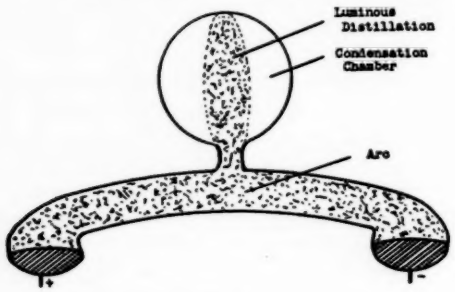


Fig. 1.

of its luminosity could be diminished by having the vapor move through an independent electric field. This vapor was later studied by Mathies[3] and by Child.[4] The fact that the luminosity may be quenched by means of an electric field indicates that the luminosity is intimately connected with the ions present in the vapor. Just how these ions are responsible for the luminosity is a more difficult question to answer.

A series of very careful, important, and interesting experiments have been done by Strutt[5] on this vapor, as well as the vapors from other substances in the electric discharge. Strutt[6] has shown that, if the vapor is allowed to flow past the electrodes of an independent electric circuit (as in Figs. 2 and 3), the luminosity of the vapor stops abruptly at the cathode, regardless of whether the anode or the cathode is upstream. That is to say, the luminosity is unaffected upon passing the anode, but is quenched upon passing the cathode. Moreover the current in this

[1] Phys. Zeit., Vol. 4 (1903), p. 321.
[2] Phys. Zeit., Vol. 4 (1903), p. 440.
Ann. d. Phys., Vol. 14 (1904), p. 520.
Ann. d. Phys., Vol. 18 (1905), p. 231.
[3] Ann. d. Phys., Vol. 30 (1909), p. 633.
Verh. d. Deut. Phys. Ges., Vol. 12 (1910), p. 754.
[4] PHYS. REV., Vol. 22 (1906), p. 221.
Phil. Mag., Vol. 36 (1913), p. 906.
PHYS. REV., Vol. 4 (1914), p. 387.
[5] Proc. Roy. Soc., Vol. 88 (1912), p. 110; Vol. 90 (1914), p. 364; Vol. 91 (1914), p. 92.
[6] Proc. Roy. Soc., Vol. 91 (1914), p. 92.

independent circuit depends only upon the position of the cathode—regardless of the disposition of the anode. Strutt reasons in the following manner. Since the luminosity stops abruptly at the cathode and does not reappear after the vapor has passed out of the electric field, the quenching of the luminosity must be due to the removal of the positive ions at the cathode. An equal number of negative ions must be removed at the anode or else the independent electrical circuit would continually charge up. The removal of these negative ions at the anode (Fig. 3) does not apparently affect the luminosity. Now, if the light were pro-

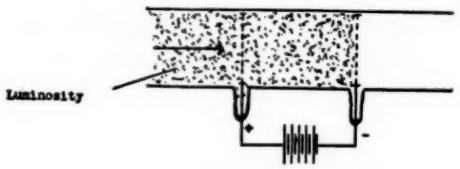


Fig. 2.

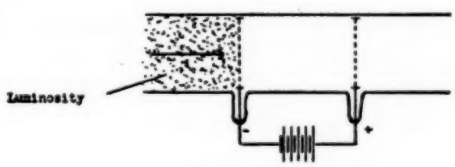


Fig. 3.

duced by the recombination of positive and negative ions, the rate of production of light would be proportional to the rate of recombination of ions, which, in turn, is proportional to the product of the numbers of positive and negative ions per unit volume of the vapor. Thus the removal of either positive or negative ions should stop the luminosity. Since this is not observed, Strutt comes to the conclusion that the positive ions themselves emit the light as positive ions—the radiation being started while the ions are in the arc. This necessitates that the emission last during the time the vapor takes in moving from the arc to the place of observation, a time which Strutt[1] calculates to be of the order of one thousandth part of a second, which is not to be expected if the electrons in the atom radiate according to Lorentz's calculations. Lorentz[2] calculates that an electron radiating light of wave-length $6. \times 10^{-5}$ cm. should lose amplitude in the ratio $1/e$ in $4. \times 10^{-8}$ secs., and could not therefore carry its luminosity with it for any appreciable distance. However as Strutt[3] remarks, if the radiation is due to a uniformly revolving ring of equally spaced electrons, the system will radiate its energy very much more slowly than a single electron, and might therefore travel a considerable distance before extinction.[4] Strutt also suggests that perhaps the positive ion, while in the arc, is set into a sort of unstable state, such that it may later emit radiation by some "hang-fire" mechanism—the

[1] Loc. cit.
[2] Theory of Electrons (1909), p. 259.
[3] Loc. cit.
[4] Loc. cit.

actual radiation taking place over a very short interval of time, during which the ion does not travel an appreciable distance. That is to say, a positive ion leaves the arc with a certain amount of stored up energy, but does not radiate. Due to its critical state it may radiate later. After it has moved some distance, it emits its radiation in a sort of flash which indicates its position at the time. (The length of time before the light is emitted might be expected to be controlled by the laws of probability.) Thus the numerous ions emitting light after different intervals of time have elapsed, give an apparently continuous decay in luminous intensity along the column of distilled vapor. This might be expected on the Bohr theory of the atom. In any case, because of the electrical properties of the vapor, as cited above, Strutt holds firmly to the view that the light is emitted by the positive ions as positive ions and rejects the recombination theory.

Child[1] criticises these conclusions of Strutt by suggesting that, while the total number of negative ions (see Fig. 3) may be decreased by the removal of a fraction of them at the anode, the density of the ions in the space between the electrodes will be unchanged if the velocity of the ions in this space is decreased by the same fraction. The number of ions passing through unit area perpendicular to the stream per second is the product of the density of the ions by their velocity. If this number of ions is altered by a given factor, the density of the ions will be unchanged if their velocity is altered by this same factor. Moreover the direction of the electric field is such as to oppose the motion of the negative ions, and so decrease their velocity.

Child[2] describes an experiment in which the luminosity is diminished in the neighborhood of an electrode charged positive with respect to the arc circuit, but offers no explanation of the phenomenon. He also notices that, if the potential of this electrode is sufficient, the luminosity is increased, being now of a greenish color. Later, by means of an alternating current arc and a stroboscopic wheel, Child[3] finds that the luminosity persists for about a thousandth of a second after the current has ceased. Child[4] again criticises Strutt's work in a later paper. In another paper Strutt[5] presents evidence to the effect that the emitting centers are negatively charged particles, in contradiction to his previous conclusions.

In view of this difference in opinion as to the manner in which this

[1] Phil. Mag., Vol. 31 (1916), p. 139.
[2] PHYS. REV., Vol. 4 (1914), p. 387.
[3] PHYS. REV., Vol. 9 (1917), p. 1.
[4] Phil. Mag., Vol. 37 (1919), p. 61.
[5] Proc. Roy. Soc., Vol. 94 (1918), p. 88.

luminosity is produced, it was thought advisable to investigate this luminosity further, with the view of deciding between several possible sources of the radiation.

II. Experimental Arrangements.

In the investigation of this vapor, the first apparatus used is shown in Fig. 4. In this apparatus (made of glass) the arc was maintained between the mercury pool, K, and the cylindrical anode, An, made of nickel, and sealed into the glass by means of a platinum wire. The vapor from the arc passed through this anode and into the larger tube in which the four auxiliary electrodes A, B, C, D, made of a nickel wire grid of rather coarse mesh, were sealed. These electrodes were made by

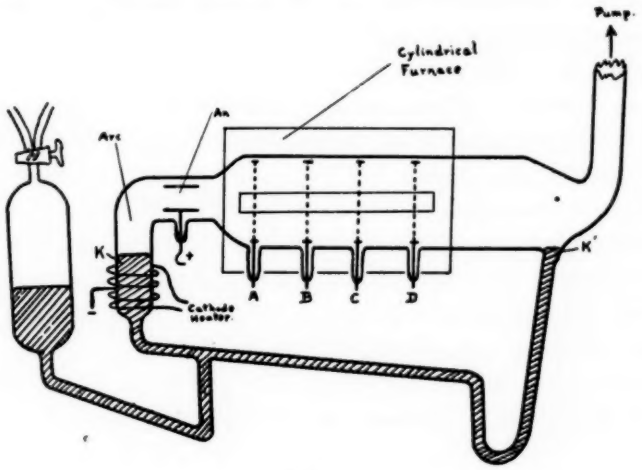


Fig. 4.

stretching fine nickel wires across a nickel ring 3 cms. in diameter. The wires were all parallel, and spaced about 2 mms. apart. The vapor condensed at the end of the tube at K', and, by means of the connecting tube, flowed back to the cathode, K. A mercury reservoir was connected to this system, so that by varying the pressure of the air over the mercury, the level of the mercury could be altered. The wide tube, containing the auxiliary electrodes, was surrounded by a cylindrical electric heater to prevent the mercury condensing on the walls of the tube. A slit about one centimeter wide was cut along the side of this heater parallel to its axis, through which the luminosity of the vapor could be observed. The tube in which the arc was maintained was about 2 cms. internal diameter, and the arc was about 10 cms. long. The large tube was 3.7 cms. internal diameter and 23 cms. long. The heated portion of this

tube was 16 cms. long, and the auxiliary electrodes were spaced 3.5 cms. apart. A coil of nichrome wire was wound about the mercury cathode so that it could be heated, and the vaporization of the mercury thereby controlled. The apparatus was evacuated by means of a combination motor-driven oil pump and Langmuir condensation pump made by the General Electric Company. With this pump an extremely high vacuum could be obtained. The arc was started by heating the cathode until the mercury was vaporizing rapidly, and a discharge from an induction coil was then passed between the anode, An, and the electrode, A. The arc was maintained by a battery of Edison storage cells, giving about 80 volts, with a suitable resistance in series. An ammeter was inserted to measure the current through the arc, and a voltmeter was placed across the arc to measure the potential drop across the arc. The arc had a great tendency to form between the anode, An, and the mercury pool, K', but, by repeated trials, the arc could be made to "strike" in the proper manner. When running steadily with a current of three amperes, the drop in potential across the arc was about 12.9 volts.

When the arc is in operation, the vapor pressure in the arc is quite appreciable, and causes some of the vapor to be blown through the cylindrical anode, An, into the wider tube in which the auxiliary electrodes are placed. As the vapor flows down this tube, it can be clearly seen by its reddish luminosity, for the entire length of the tube. The luminosity is, under proper conditions, quite bright, and its reddish color stands out in sharp contrast to the light from the arc itself, which is of a bluish color. However, when examined with a small pocket spectroscope, it is seen that the spectra are the same, except for the relative intensities of the lines. If the cathode heater is stopped, so that the cathode may cool down, the luminosity of the distilled vapor becomes very faint. Thus it seems that it is necessary to supply more mercury vapor than the arc normally forms, if the luminosity is to be at all intense. Keeping the current through the arc constant, as the cathode is heated, so as to supply more vapor to the arc, the potential drop across the arc increases, and, if the cathode is very hot, so that the mercury vapor in the arc is very dense, the potential drop mounts to about 25 volts or slightly more, when the arc suddenly goes out. If the cathode heater is stopped, so that the vapor is distilling very slowly, the arc frequently goes out. Thus it seems that it is best to have the heater running for the steady maintenance of the arc as well as for producing a bright stream of distilled vapor. As well as the author is aware, this is the first time that, in such experiments, the cathode has been independently heated.

An inductance was placed in series with the arc to further steady it. It was observed that the auxiliary electrodes, A, B, C, D, tend to charge up to a negative potential of about a volt with respect to the anode, An, which was connected to earth.

The effect was tried of applying various potentials to the auxiliary electrodes.

III. The Positive Effect.

If any one of the electrodes is made positive, and this positive potential slowly increased, while all other electrodes are left insulated so as to take up what potential they may, the following effects are observed. First there is a quenching of the luminosity near the electrode on the upstream side, which quenched region broadens, reaching further and further upstream until the luminosity in almost the entire region between the electrode and the arc's anode, An, has been quenched. While this has been going on, a quenching of the luminosity near the electrode on the downstream side is noticed, and, as the potential is increased, this region of quenching reaches further and further downstream. Thus there is a quenching of the luminosity in the neighborhood of the electrode and on both sides of it. All of these effects have occurred at potentials up to two volts. At two volts the luminosity in the region between the electrode and the anode, An, has become too faint to observe. It may be mentioned here that the boundary of the quenched region is not at all sharp, but is a gradual shading off. If the potential is further increased a bluish glow starts on the upstream side of the electrode, and reaches further and further upstream as the potential is increased, becoming brighter all the while. Also, during this process, the bluish luminosity passes through the electrode, and, as the potential is increased, reaches further and further downstream. The important thing to notice is that the reddish luminosity is diminished on both sides of the electrode, but is not permanently quenched, for it reappears further down the tube. Moreover the bluish luminosity which in all probability is due to ionization by collision, occurs on both sides of the electrode, in its immediate neighborhood. Both of these effects may be easily explained in the following manner. As the potential of the electrode is increased, the electric field on the upstream side is increased, and the electrons in the vapor are accelerated. As the velocity of the electrons is increased the density is accordingly diminished and (assuming that the light is produced by the recombination of positive and negative ions) the luminosity is therefore decreased. As this increased velocity probably persists for some time after the electrons have left the field, the quenching on the downstream side is explained. As the electrons gradually lose this increased

velocity the density increases, and the luminosity is restored. If, during this procedure, the velocity of the electrons relative to the vapor has reached a sufficient value to ionize, the production of the bluish glow is explained—the glow marking the region in which the velocity of the electrons relative to the vapor is equal to or greater than the ionizing velocity. This explanation of the quenching may be objected to on the grounds that the number of electrons going to the electrode is greatly increased by the electric field drawing a large number of ions out of the arc, which, as we shall seel ater, is most certainly the case. However the increase in velocity under the electric field is of higher order than the

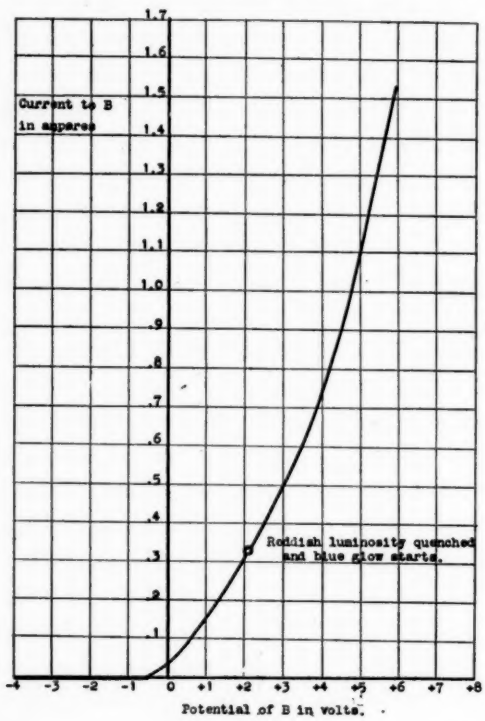


Fig. 5.

increase in the number of electrons, so that the explanation still holds. This will be discussed again later. This positive effect is, in all probability, the effect observed by Child[1] when the auxiliary electrode is made positive with some part of the arc circuit. The current to this electrode is rather large, as may be seen from the curve in Fig. 5, which shows the variation of the current to the electrode with the potential of the electrode. The current for five volts is seen to be over one ampere. As

[1] PHYS. REV., Vol. 4 (1914), p. 387.

will be seen later, this is very much larger than when the electrode is negative.

IV. THE NEGATIVE EFFECT.

If any one of the auxiliary electrodes is given a negative potential, and this potential is slowly increased (negatively), while all other electrodes are left insulated so as to take up what potential they may, it is found that the luminosity on the upstream side is unaffected even with potentials as high as 300 volts, except within about half a millimeter of the electrode. However the luminosity on the downstream side is diminished by this procedure. If the density of the stream of distilled vapor is not too great, the luminosity on the downstream side may be, as well as can be judged, completely quenched. The potentials required for this effect vary greatly with the density of the vapor stream. If the density of the vapor stream is very small, so that the luminosity is

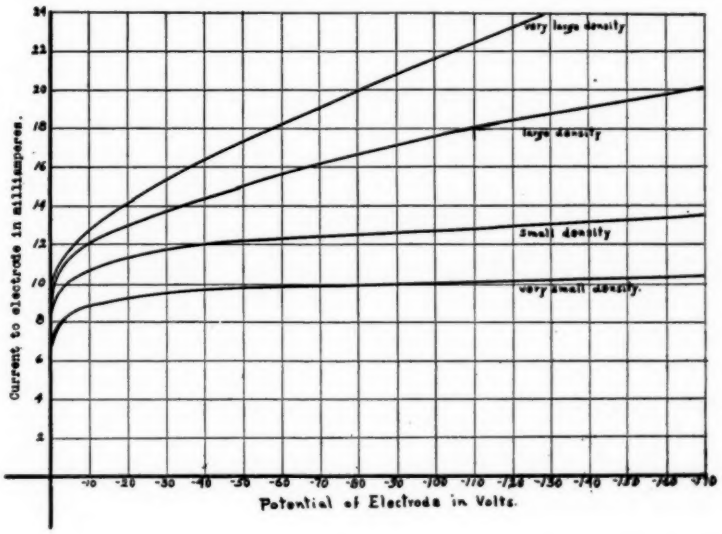


Fig. 6.

feeble, small potentials of the order of one to three volts are sufficient to quench the luminosity. On the other hand, if the density of the vapor stream is large, potentials as high as 300 volts will not completely quench the luminosity. This seems to run parallel with the current to the electrode. In Fig. 6 is shown the way in which the current to the electrode varies with the potential of the electrode for negative potentials. From these curves it may be seen that, if the vapor density is small, a few volts seem to be sufficient to saturate the current. However, as the density of the vapor increases, it becomes more and more difficult

to obtain saturation, and the curve then resembles the curves obtained in the conductivity of flames. As the potential difference is increased the current continues to increase and the luminosity continues to decrease. However the current never becomes saturated and the luminosity is never completely quenched. This behavior of the current with respect to the potential is, in all probability, to be explained in the same manner as in the case of flames, by the large drop in potential in the neighborhood of the cathode. This is supported by the observations of Strutt[1] to the effect that almost the entire drop in potential occurs within a region very close to the cathode. It is to be expected that, as the density of the vapor increases, there are more ions swept out of the arc by the vapor stream, and as the density of the ions increases the effect should become more prominent. From Fig. 6 it will be noticed that the currents obtained in the negative effect range from ten milliamperes or lower to about twenty-five milliamperes. This is about one hundredth of the current obtained in the positive effect.

If the electrode, *B*, in Fig. 4, is maintained at a certain negative potential, with a milliammeter in series to measure the current, and the electrode, *A*, is then given a negative potential, the current to *B* is seen to decrease. This effect is much more marked when the density of the vapor is small. If the vapor density is large, the application of a negative potential to *A* does not appreciably affect the current to *B*. This is what we should expect from the above consideration of the currents to the electrodes, for with a dense vapor flowing past the electrode, *A*, only a very few positive ions are removed, and so the number of ions reaching the electrode, *B*, is practically the same as when *A* is uncharged. The important thing to notice in the case of the negative quenching is that, once the luminosity has been quenched, its quenching is permanent. That is to say, the luminosity does not reappear after the vapor has passed out of the electric field. This, together with the parallel relations of the quenching of the luminosity and the current to the electrode, seems to indicate that the quenching of the luminosity is dependent, in some manner, upon the removal of the positive ions at the negative electrode.

V. THE STRUTT ARRANGEMENT.

In Strutt's[2] experiments the distilled vapor was allowed to flow past the electrodes of an independent circuit, insulated from the arc circuit, as indicated in Figs. 2 and 3. These experiments of Strutt's have been discussed to some extent in the introduction. Strutt[1] found that, as the

[1] Proc. Roy. Soc., Vol. 91 (1914), p. 92.

[2] Loc. cit.

vapor flowed past the cathode of an auxiliary circuit, it had part of its luminosity removed; while, as it flowed past the anode, no change in luminosity was observed. Also the current in the auxiliary circuit is determined by the potential difference between the electrodes, and the position of the cathode. That is to say, regardless of the position of the anode, the current for a given cathode is the same. Also, if the vapor is made to flow past two such independent circuits in succession, increasing the current in the first circuit by increasing the potential difference between the electrodes of that circuit, decreases the current in the second circuit as is to be expected, as then more ions are removed by the first circuit, and so fewer reach the second. All of these experiments of Strutt's have been repeated by the present writer and his observations verified. Moreover the current in the auxiliary circuit seems to be exactly the same, for a given potential difference, as the current in the circuit formed by the same electrode as cathode, and the anode of the arc itself as anode. That is to say, the current is the same in the Strutt arrangement as in the negative effect, if the electrode used in the negative effect is the same electrode as the one used as cathode in the Strutt arrangement. Also, as well as can be judged, the quenching of the luminosity in the Strutt arrangement is the same as in the negative effect. It may be well to remark here that the presence of the mercury pool, K', at the end of the tube and in metallic connection with the cathode of the arc, does not seem to interfere with any of the electrical effects, either positive effect, negative effect, or the Strutt arrangement; for the connecting tube was later removed and all previously described work repeated. No differences in the results were noticed. From the shape of the curves in Fig. 6 it does not seem that an appreciable amount of ionization by collision goes on in connection with the negative effect. This is to be expected, as the large drop in potential occurs only in the immediate neighborhood of the negative electrode, and only the positive ions, which have small mobilities, are accelerated here. This also is the case with the Strutt arrangement. Occasionally, however, in working with the negative effect, ionization by collision does become appreciable, and then an arc forms. Whenever this happened the circuit was quickly broken to extinguish the arc and the circuit reformed. This does not happen often, however. Whenever the arc forms the color of the light is bluish, quite different from the reddish luminosity of the distilled vapor. An arc was never formed in working with the Strutt arrangement.

VI. Discussion of the Electrical Effects.

The positive effect has already been discussed in some detail and an explanation of the effect given. There are, however, a few points of difficulty. In the first place we assumed that the negative ions are accelerated as they approach the positive electrode. By this we mean merely that they have their velocities relative to the vapor increased. Of course the electrons are suffering collisions during their motion, but their velocities relative to the vapor will be the product of their mobilities into the strength of the electric field. In order that this average velocity of the electrons should increase it is necessary that the electric field near the positive electrode be stronger, which is not to be expected if a large drop in potential occurs at the negative electrode. Now it is found in the conductivity of flames that placing an easily ionized salt on the cathode increases the current enormously, and makes the potential gradient more nearly uniform along the flame. In the positive effect the cathode of the auxiliary circuit is the anode of the arc itself. This anode, being in the arc itself, is surrounded by intensely ionized vapor. This should produce the same effect as a salt on the cathode of a circuit through a flame. Thus it is reasonable to suppose that, in this case, the largest drop in potential should occur at the auxiliary electrode used in the positive effect. Now it may be seen by comparing currents in the positive and negative effects, that the currents involved in the positive effect are from twenty to one hundred times the currents involved in the negative effect. This increased conductivity must certainly be due to the cathode of the auxiliary circuit (anode of the arc) being surrounded by intensely ionized vapor. Now from later considerations it follows that there are 1.51×10^{16} molecules of mercury vapor per cubic centimeter in the distilled vapor. This corresponds to the number of molecules per cubic centimeter of any gas at 0° C. and at a pressure of .418 mm. of mercury. Now if we assume that, when the positive quenching occurs, we have an electric field intensity of one volt per centimeter, the value of X/p is 2.4. Townsend[1] gives a series of values for the velocities of electrons in air at low pressures as a function of X/p. For $X/p = 2$, $U = 1.75 \times 10^6$ centimeters per second. For $X/p = 5$, $U = 3 \times 10^6$ centimeters per second. Now the velocity of electrons through mercury vapor might be expected to be smaller than through air, because the mercury atom is larger than the atoms making up air. However the mercury molecule consists of a single atom, while the molecules in air consist of two atoms, and so there is probably not a great amount of difference between the sizes of the mercury and air molecules. It is

[1] Electricity in Gases (1915). p. 124

then probably safe to assume that the velocities of electrons in mercury vapor are the same as in air: at least they are of the same order. If we interpolate between Townsend's values of the velocity we have $U = 2 \times 10^6$ centimeters per second, approximately. Initially the velocity of the electrons is the same as the velocity of the distilled vapor, which, from later considerations, is 3,740 centimeters per second. When the electric field has been established the velocity of the electrons is 2×10^6 centimeters per second. This means that the velocity has been increased 535 times. Now by a comparison of the currents obtained in the positive and negative effects it is seen that the current in the positive effect is about thirty times the current in the negative effect, when the luminosity has been quenched. Since the current in the negative effect is a measure of the total number of positive ions flowing past per second (when the vapor density is not too large), and therefore a measure of the number of electrons when no electric field is acting (since there must then be equal numbers of positive and negative ions), it follows that, by virtue of the electric field, the total number of electrons flowing has been increased thirty times, while the velocity has been increased 535 times. The density of the electrons has then been decreased about 18 times. As the number of positive ions is probably not altered, due to their small mobilities, the luminosity should, as a first approximation, be proportional to the density of the electrons. Therefore the intensity of the luminosity should be decreased about 18 times. This explains the decrease in luminosity on the upstream side of the electrode. In regard to the diminution of the luminosity on the downstream side there are several possibilities.

In the first place, if the mean free paths of the electrons in the vapor are sufficiently large, the electrons which do not strike the electrode (which are equal in number to the positive ions which pass through the electrode) will continue with their high velocities for a certain distance before their increased velocities are lost by collisions with the vapor Jeans[1] gives a series of numerical values of the mean free paths of molecules in various gases. His table does not include mercury vapor, but, for the heavy gases, it may be seen that the values are not widely different. The value given for xenon is 3.5×10^{-6} centimeters at 0° C. and 760 mms. pressure. Now both xenon and mercury vapor are monatomic gases, and their atomic weights do not differ so very widely. We can therefore take the mean free path of mercury atoms in mercury vapor to be 3.5×10^{-6} centimeters at 0° C. and 760 mms. pressure (if such were possible). The density of the vapor in the experimental tube was 1/1,816

[1] Dynamical Theory of Gases (1916), p. 341.

of the value under standard conditions. We must therefore increase the mean free path, λ, 1,816 times (since λ is inversely proportional to the density) so that we have $\lambda = .0545$ millimeters. This is for mercury atoms. For electrons this value must be increased by the factor $4\sqrt{2}$, which gives $\lambda = .309$ millimeters. This means that, in going one centimeter, the electron must make about 32 collisions. So, unless the electron could suffer a number of collisions without having its velocity and direction sensibly altered, we should not expect the quenching to extend this far. There are, however, several ways in which this region of quenching could be increased.

1. A considerable fraction of the electrons, of a given mean free path, will have actual free paths several times the mean free path. This should lengthen the region and produce a gradual shading off of the luminosity, as is actually observed.

2. When the velocity of an electron relative to the vapor is large it is difficult for recombination to occur, and, if the velocity is not too high, the collisions between the electrons and molecules of mercury vapor will be almost perfectly elastic. Therefore, while collisions will deflect the path of the electron in all possible directions, so that its velocity relative to the vapor is zero, still the numerical value (regardless of direction) of its velocity will decrease very slowly. By the time its velocity has dropped to a value at which recombination can take place, it has been carried along by the vapor as it flows along the tube. (This effect might also occur to some extent on the upstream side.)

3. If the electric field on the upstream side of the electrode increases near the electrode there must be an excess of negative charge in the vapor. The vapor is therefore acted upon by a force in the direction of the electrode—resulting in the vapor as a whole being accelerated, causing accordingly a decrease in density of both positive and negative ions and thereby a decrease in intensity of luminosity.

4. The absolute temperature in the experimental tube is probably twice the absolute temperature at 0° C., and this should cause a slight increase in the mean free path.

5. It is perhaps not unreasonable to suppose that, at high velocities, the electrons may collide with the atoms without being greatly deflected, for electrons of much higher velocities (β rays) most certainly do. This would have the same effect as lengthening the free paths of the electrons.

These five explanations are all in the right direction to explain the large region of downstream quenching. They probably all do help to increase this region—the extent to which they actually contribute is however debatable. At any rate, the downstream quenching is decidedly

real, and the quenched region extends downstream from a few millimeters to two or three centimeters. It is difficult to explain the positive effect on the theory that the positive ions are the light emitters, as positive ions.

The negative effect is easily explained on either theory. If the light emitters are the positive ions themselves, the removal of the positive ions at the electrode should quench the luminosity. If the light is due to the recombination of positive and negative ions, the removal of the positive ions at the electrode will prevent them later recombining with negative ions, and so stop the luminosity.

The only effect that seems at all to stand in the way of the recombination theory is the quenching of the luminosity with the Strutt arrangement. Referring again to Figs. 2 and 3, it is easy enough to explain the case (Fig. 2) where the negative electrode is upstream on either theory, for the removal of the positive ions should quench the luminosity. In the case shown in Fig. 3, however, the two theories, on first consideration, predict different effects. On the theory that the light is given out by the positive ions we should expect that the luminosity should be unaffected until the vapor has reached the negative electrode, where the positive ions are removed and the luminosity accordingly quenched, as is observed. However, on the recombination theory, the removal of either positive or negative ions should diminish the luminosity. Since the number of electrons removed at the positive electrode must be equal to the number of positive ions removed at the negative electrode, we should expect that the luminosity should be diminished at the positive electrode when the electric field is established. This, however, is not observed. The conclusions drawn from this by Strutt have already been discussed, as well as Child's criticism of them. It appears to the present writer that Child's criticisms are just. Due to the high mobilities of the electrons at low pressures it is possible for their velocities to be very greatly altered by the electric field, even when it is weak. This will tend to prevent a decrease on the density of the electrons. We can show that the densities of the positive and negative ions must be very nearly equal in the following manner.

Suppose that all of the electrons were removed at the upstream anode. The postive ions will then continue on their path until they reach the cathode. The density of the positive ions in the space between the electrodes will probably then not vary greatly from point since recombination cannot occur. Let there be n ions per cubic centimeter in this region, and let e be the charge on each. Then if X be the electric field, and x the distance from the anode, measured toward the cathode, we

will have

$$\frac{dX}{dx} = 4\pi ne,$$

and so

$$X = 4\pi nex + X_0$$

and

$$V = \int 4\pi nex \cdot dx + \int X_0 \cdot dx$$

or

$$V = 2\pi nex^2 + X_0 \cdot x + V_0,$$

where V is the potential difference between the anode and the point x. Let us take $V_0 = 0$, and let $x = a$ at the cathode. Also suppose that X_0 is negligible. Then

$$V = 2\pi nea^2.$$

Now neu is of the order of the maximum current involved in the experimental work, and is about 10 milliamperes across a cross section of about 10 square centimeters, or one milliampere across unit area. Therefore

$$ne = \frac{i}{u} = \frac{3 \times 10^6}{3{,}740} \text{ E.S.U.} = 8 \times 10^2 \text{ E.S.U.},$$

since the velocity, u, of the positive ions is the same as the velocity of the vapor, which, from later considerations, is shown to be about 3,740 centimeters per second. Since $a^2 = 10$, this gives

$$V = 5 \times 10^4 \text{ E.S.U.}$$

or 1.5×10^7 volts.

The actual potential difference between the electrodes is of the order of 50 volts, however. To give this value ne should be 2.7×10^{-3} E.S.U, or 3.3×10^{-6} of the value obtained on the assumption that no electrons entered the space between the electrodes. That is to say, the value 2.7×10^{-3} E.S.U. represents $(n_1 - n_2)e$, where n_1 is the number of positive ions per cubic centimeter, and n_2 is the number of electrons per cubic centimeter.

We then have

$$\frac{(n_1 - n_2)}{n_1} = 3.3 \times 10^{-6},$$

which shows that the number of positive ions per cubic centimeter and the number of electrons per cubic centimeter are very nearly equal. If n_1 and n_2 are nearly equal it follows that the velocities of the electrons must be decreased by the electric field. Now a very weak field is sufficient to make a great change in the velocities of the electrons, as stated before. This can be shown in the following manner. The initial velocity of the electrons is the velocity of the vapor, 3,740 centimeters

per second. The velocity of the electrons relative to the vapor is the product of the mobility of the electrons into the electric field. As the field tends to retard the electrons, the actual velocity of the electrons will be $(u - kX)$ where k is the mobility. This velocity can thus be reduced to zero by a field of .0018 volts per centimeter, if we assume that the mobility of the electrons is 2×10^6 centimeters per second for a field of one volt per centimeter. The electrons are not brought completely to rest, and so the actual field must be less than this. This is in agreement with Strutt's observations that almost the entire drop in potential occurs within a region very close to the cathode. Thus, in spite of the large drop in potential at the cathode, the weak field between the electrodes would be sufficient to slow down the electrons so that their density is unchanged. With these considerations, then, the quenching with the Strutt arrangement can be explained on either theory.

In the case in which the cathode is upstream, the electrons will be accelerated by the electric field, and their density will accordingly decrease about one thousand times. Thus if only one thousandth of the positive ions reaching the cathode enter the region between the electrodes the densities of positive and negative ions in this region will be equal. The luminosity will then be reduced to one millionth of its previous value.

The conclusions to be drawn from the electrical effects are these. The negative effect and the quenching of the luminosity with the Strutt arrangement can be explained on either theory. The positive effect can be explained very easily on the recombination theory, but an explanation on the other theory does not immediately present itself.

VII. Relative Quenching of the Lines in the Visible Part of the Spectrum.

The following lines may easily be observed in the spectrum of the distilled vapor:

* yellow $\lambda = 5790$. A.U.
* yellow $\lambda = 5769$. A.U.
* green $\lambda = 5461$. A.U.
* violet $\lambda = 4358$. A.U.
violet $\lambda = 4078$. A.U.
violet $\lambda = 4047$. A.U.

The lines marked with an asterisk are the most prominent, and are accordingly the ones upon which observations were made. In addition to these are two rather faint orange lines and a somewhat faint greenish-blue line.

These lines are the same lines as occur in the spectrum of the mercury

arc itself, the only difference in the spectra being in the relative intensities of the lines.

The question that immediately arises is whether all lines are quenched in the same ratio, or whether some lines are more easily quenched than others. To be sure the color of the luminosity, as observed by the eye, does not seem to change when the luminosity is diminished by the electric field. Strutt also is of the opinion that the lines are quenched in the same ratio, but no quantitative measurements to determine this point were made by him. The present writer has measured the relative intensities of the four most prominent lines before quenching to their intensities after the application of the electric field, by means of a Nutting polarization photometer in connection with a Hilger wave-length spectroscope. The conclusions reached in these experiments were that all lines are quenched in the same ratio; or, if not, the differences are very small.

VIII. Intensities of the Visible Lines in the Spectrum of the Bluish Glow as Compared with Those in the Luminosity of the Distilled Vapor.

When observed with a spectroscope the light from the bluish glow shows the same lines as the reddish luminosity of the distilled vapor. However, since there is a marked difference in the color of the two types of luminosity as viewed by the eye, one would expect that this difference is due to a difference in the relative intensities of the lines in the two cases. These relative intensities have been measured by means of the Nutting polarization photometer in connection with a Hilger wave-length spectroscope. The results of these experiments showed that, when the yellow line $\lambda = 5{,}790$ was of the same intensity in the two cases, the yellow line $\lambda = 5{,}769$ was about six per cent. more intense in the blue glow than in the reddish luminosity, the green line $\lambda = 5{,}461$ was about 76 per cent. more intense in the blue glow, and the violet line $\lambda = 4{,}358$ was about three times as intense in the blue glow as in the reddish luminosity of the distilled vapor. It seems then that the process producing the blue glow is different from the process concerned in the production of the persistent reddish luminosity.

IX. The Addition of Un-ionized Mercury Vapor to the Luminous Vapor.

The previously described experiments on the electrical quenching of the luminosity show that the luminosity is intimately connected with the positive ions present in the vapor, so that the removal of the positive ions stops the luminosity. This would indicate that the light is not produced

qy the influence of the intense light from the arc, for if this were the case the luminosity should occur beyond the electrodes used in the negative quenching effects. It was thought, however, that this point might well be investigated in a different manner. Accordingly the apparatus shown in Fig. 7 was used. In this apparatus, also made of glass, the arc was main-

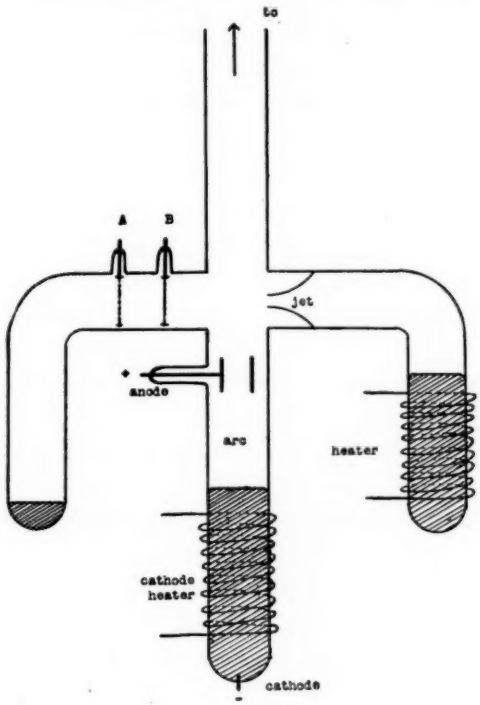


Fig. 7.

tained between the mercury cathode and the cylindrical anode made of nickel. The cathode was provided with an independent electrical heater so that the volatilization of the mercury could be controlled. The vapor from the arc moves up the vertical tube leading to the pump. Connected with this tube on one side was a tube containing mercury which could be heated by an independent electric heater. The mercury vapor from this tube passes through a glass jet and is blown across the main tube where it is exposed to the direct action of the light from the arc itself. It then passes into the side tube at the left where it finally condenses. Grids of fine nickel wire forming the electrodes *A* and *B* were placed in this tube so that the vapor passing through them could be studied.

The arc was run with about 2.4 amperes with the heater only hot enough to prevent the arc going out. In this condition the luminosity in the

vertical tube above the anode was too faint to observe. The part of the tube in which the arc was formed was wrapped with asbestos, as was the first experimental tube (Fig. 4), to prevent the brilliant light from the arc interfering with observations. When the mercury in the side tube was heated, the mercury vapor was forced through the jet and into the side tube to the left where it could be seen condensing. A very faint luminosity appeared in this tube in the region near the electrodes *A* and *B*. This luminosity was too faint for its color to be determined, but it was found that it could be quenched by establishing a difference in potential between the electrodes *A* and *B*. It is highly probable that this luminosity is due merely to the horizontal stream of mercury vapor carrying with it some of the luminous vapor which has distilled away from the arc. This vapor was too faint to observe when it was in the vertical tube, for there was considerable light from the arc itself which had been reflected from the walls of the tube. This light is probably much brighter than the faint luminosity of the distilled vapor, and so the latter is obscured. However as soon as this vapor has been carried into the side tube where it is fairly dark, its luminosity is more easily observed.

When the cathode of the arc was strongly heated, a very bright column of distilled vapor could be seen in the vertical tube. In this case, when the un-ionized mercury vapor was forced through the glass jet, the resulting stream of un-ionized vapor was deflected upwards and carried along by the current of luminous vapor. The stream of un-ionized vapor could be traced through the luminous vapor by its lack of luminosity. The effect is illustrated in the adjoining figure (Fig. 8). This seems to show that the intense light from the arc itself does not produce any great amount of luminosity in un-ionized mercury vapor.

Fig. 8.

In this connection the dependence of the intensity of the luminosity in the distilled vapor upon the rate of volatilization of the mercury at the cathode of the arc is worthy of consideration. Due to the more rapid volatilization of the mercury the potential drop across the arc increases and, in consequence, the energy expended in the arc increases. If this energy is used up in ionization we should expect the number of ions to be increased. However the potential drop across the arc is increased by only a few per cent. by the increased vaporization of the mercury, and as the intensity of the luminosity is increased many times, we should not expect that the corresponding increase in ionization should account for the increase in luminosity. The part played by this increased

supply of mercury vapor is probably to place a larger number of neutral molecules between the positive and negative ions, and so diminish their rate of recombination. This permits them to travel a greater distance away from the arc before recombining, and thereby losing their ability to produce light. When very little vapor is distilling away from the arc there are not many neutral molecules between the ions, and so the ions recombine quickly. Accordingly they have become very few in number before they have left the arc for any appreciable distance. The change in the velocity by the different rate of volatilization and also the corresponding alteration in the densities of the ions contribute to the effect.

X. The Velocity of the Vapor.

In all of these experiments with the luminosity of the distilled vapor it is highly desirable to know the velocity with which the vapor flows along the tube. The velocity of this vapor was calculated by Stark[1] from the potential gradient set up in the vapor due to a magnetic field perpendicular to the direction of flow. The value given by Stark is 28,000 centimeters per second. Strutt[2] calls attention to the fact that the velocity of the vapor cannot exceed the molecular velocity of agitation at the temperature of the lamp. Taking this temperature to be 273° C. Strutt gives 50,000 centimeters per second for the velocity of molecular agitation and therefore sets this as an upper limit for the velocity of the vapor.

It was thought highly desirable to attempt a direct measurement of the velocity of the vapor. Accordingly the first apparatus (Fig. 4) was arranged as illustrated in Fig. 9. The anode of the arc was earthed and an audion oscillator arranged so as to cause the potential of one of the electrodes (B) to vary harmonically from zero to about 100 volts negative potential. In this way the electrode would alternately stop the luminosity and let it through. Accordingly "puffs" of luminosity were sent down the tube with the velocity of the vapor. The tube was then observed through the holes of a stroboscopic wheel. When the stroboscopic wheel was run at such a speed that the frequency of intermittance of vision was the same as the frequency of the audion oscillator the "puffs" of luminosity appeared to remain stationary so that the distances between them could be measured. If n is the frequency of the oscillator and λ the distance between similar parts of two successive puffs, then the velocity, U, of the vapor in the center of the tube is given by

$$U = n \cdot \lambda.$$

[1] Phys. Zeit., Vol. 4 (1903), p. 440.

[2] Proc. Roy. Soc., Vol. 91 (1914), p. 92.

The puffs as experimentally observed appeared to be equally spaced, showing that the velocity of the vapor did not change rapidly as the vapor moved along the tube. However only three or four puffs could be seen at one time, and so this point deserves further consideration. The frequency of the oscillator was adjusted until λ was 2.0 centimeters.

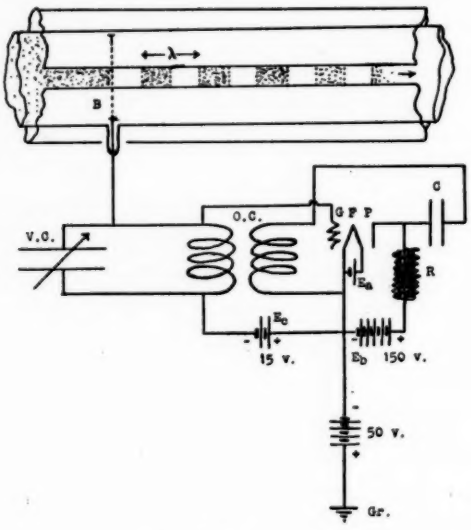


Fig. 9.

Potential of electrode *B* made to fluctuate between 0 and −100 volts by means of the audion oscillator. Luminosity observed through a stroboscopic wheel with frequency of intermittance same as frequency of oscillator. Luminous puffs then appear to stand still and can be measured.

The frequency of the oscillator was then found to be 1,870 vibrations per second. This gives U to be 3,740 centimeters per second.

This method of measuring the velocity could well be used in studying the way in which the velocity of the vapor varies as the vapor flows along the tube. A study might also be made by this method of the transition from viscous to turbulent flow. The advantage of this method is that the motion of the vapor can be traced visually.

In the determination of the velocity the current through the arc was 3 amperes and the cathode was just sufficiently hot to produce a good distillation of luminous vapor. This was the condition under which most of the conductivity and quenching effects were studied.

XI. The Density of the Vapor.

If we know the velocity of the vapor as it flows along the tube, and the cross-sectional area of the tube, we can calculate the density of the vapor

by measuring the rate of distillation of the mercury. To measure the rate at which the mercury distills, the original apparatus (Fig. 4) was altered as shown in Fig. 10. The tube connecting the mercury pool K' with the cathode K, was cut and separate tubes forming barometric columns were run down from each to the vessels R and S filled with mercury. In the side of the vessel S was an overflow tube which led to the beaker, T. The arc was run at a current of three amperes with the cathode heater at the same temperature as when the previous experiments were done. With the apparatus in normal operation the mercury in the vessel R was kept at a fixed level by pouring mercury in as it distilled away from the cathode. As the mercury condensed at the end of the tube at K' it flowed into the vessel S, and the overflow was caught in the beaker, T. The mercury which overflowed in a known time was caught and weighed. The rate of distillation of the mercury could then be determined. Let r be the rate of distillation of the mercury in grams per second. Then if α is the cross-sectional area of the tube and V the average velocity of the vapor over the cross-section, the density, d, of the vapor is given by

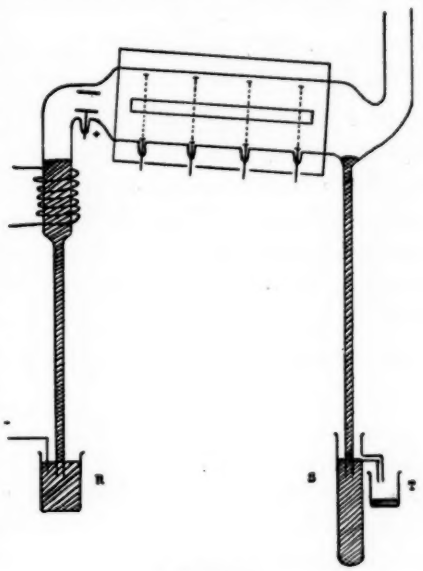


Fig. 10.

$$r = \alpha \cdot V \cdot d.$$

Now V will be $\frac{1}{2}U$ if the flow is not so fast as to be turbulent, or 1,870 centimeters per second. The diameter of the tube was 3.7 centimeters, which gives $\alpha = 10.76$ square centimeters. In one hour 356.91 grams of mercury distilled over—giving the rate, $r = .0991$ grams per second. Thus $d = 4.95 \times 10^{-6}$ grams per cubic centimeter. The mass of the mercury atom is 3.25×10^{-22} gram. There are then 1.51×10^{16} molecules of mercury per cubic centimeter of the vapor.

XII. The Nature of the Flow of the Vapor along the Tube.

It has been shown by Reynolds[1] that, when a fluid flows along a uniform circular tube, the flow will be governed by the laws of steady

[1] Phil. Trans., Vol. 174: 3 (1883), p. 935.

viscous flow provided the average velocity of the fluid across a cross-section does not exceed a certain critical velocity. If the average velocity exceeds this critical velocity, the flow will be turbulent. If W is this critical velocity, W is given by

$$W = \frac{1000\mu}{a\rho},$$

where μ is the viscosity; a, the radius of the tube; and ρ, the density of the fluid. Taking the viscosity of the mercury vapor at 273° centigrade to be 5×10^{-4}, the radius of the tube 1.85 centimeters, and the density of the vapor 4.95×10^{-6} grams per cubic centimeter, we have $W = 54{,}000$ centimeters per second, which is many times our experimentally determined velocity. Thus we are safe in assuming that the flow of the vapor is governed by the ordinary viscous equations. If a gas of viscosity μ flows along a uniform circular tube of radius a and length L, then, if p_1 is the pressure at one end and p_2 the pressure at the other end, the relation connecting these quantities is

$$\mu pv = \frac{(p_1^2 - p_2^2)}{8L}(a^2 - r^2),$$

where v is the velocity of the vapor at a point where the pressure is p—remembering that pv must be constant along a stream line to prevent the accumulation of gas at a point. r is the radial distance of the point from the axis of the tube. Since we know v at some point near the beginning of the tube, and from the density and temperature can calculate p, we know pv and can take p here to be p_1. We then have

$$p_2^2 = p_1^2 - \frac{8\mu p_1 v_1}{a^2} L$$

at the center of the tube. Now a density of 4.95×10^{-6} grams per cubic centimeter at a temperature of 273° centigrade corresponds to a pressure p of .836 millimeters of mercury, or 1.116×10^3 dynes per square centimeter. So

$$p_1^2 = 1.245 \times 10^6,$$

$$v_1 = 3740,$$

$$\mu = 5 \times 10^{-4},$$

$$a^2 = 3.42.$$

So

$$p_2^2 = 1.245 \times 10^6 - 4.88 \times 10^3 L.$$

When

$$L = 0 \text{ centimeters}, \qquad p_2 = 1.116 \times 10^3.$$

When

$$L = 40 \text{ centimeters}, \qquad p_2 = 1.026 \times 10^3,$$

$$\frac{p_1 - p_2}{p_1} = \frac{.09}{1.116} \qquad \text{or 8 per cent.}$$

Thus there is not a very great decrease in density in a distance of 40 centimeters along the tube. It must also be borne in mind that a decrease in density necessitates an increase in velocity. However the velocities with which we are dealing are of the order of 1/7th of the velocity of molecular agitation, and therefore of the maximum possible velocity. This approach toward the limiting velocity should tend to prevent the velocity increasing as much as out theory indicates. Attention might well be called to the fact that the relation between the pressure and the distance along the tube is parabolic, and so if the pressure falls by 8 per cent. in forty centimeters, the fall in pressure for the first part of the tube will be less than the proportional amount given by a uniform pressure gradient.

These calculations, together with the experimental observations that the luminous "puffs," in the velocity determinations, are equally spaced should permit us to take the velocity of the vapor along a streamline as constant, without any great error. We may then use the simpler equations for the case of an incompressible liquid.

XIII. The Theory of the Decay in the Intensity of the Luminosity as the Vapor Flows along the Tube.

Let us consider a number of possible ways in which the positive ions could be responsible for the luminosity of the distilled vapor.

1. In the first case let us assume the hypothesis that the positive ions do not emit the light during their entire journey from the arc, but that they are set into some sort of unstable state by the arc, such that when they emit light it is by some "hang-fire" method. That is to say, the ions travel away from the arc without emitting light. Suddenly, while they are still positive ions, they emit light—the time during which the light is emitted being so small that the ion does not travel an appreciable distance during emission. The ions are, however, continually disappearing by recombination. The light intensity is thus proportional to the number of positive ions per cubic centimeter at any point.

2. In the second case we will take as our hypothesis that the light is produced during recombination of the positive ions with electrons to form neutral molecules. Thus the light intensity is proportional to the rate of recombination, which in turn is proportional to the square of the number of positive ions per cubic centimeter (since there are equal numbers of positive and negative ions).

3. In the third case we will take as our hypothesis that the positive ions themselves emit the light—the emission occurring continually during their journey from the arc. The number of ions, however, is continually decreasing because of recombination. Thus the intensity of the light should be proportional to $ne^{-\lambda t}$, where n is the number of positive ions per cubic centimeter, and t is the time.

These various ways in which the positive ions could produce the luminosity should give rise to different laws of decay as the vapor flows along the tube. Accordingly it was hoped that, by comparing the rate at which the luminosity decayed experimentally with the rates predicted by these several hypotheses, the correct hypothesis could be determined. For the present we shall consider only cases (1) and (2). Case (3) will be discussed in the next chapter.

Let us consider a tube of uniform circular cross-section along which the vapor is flowing. Let us suppose also that the velocity, v, of the vapor is constant at all points in the tube. Let x represent the distance of any point in the vapor from some fixed plane placed perpendicular to the axis of the tube, the distance being measured in the direction of flow of the vapor. Let there be N_0 positive ions per cubic centimeter at $x = 0$. At any point at a distance x along the tube we will have

$$\frac{dn}{dt} = -\alpha n^2, \tag{1}$$

where n is the number of positive ions per cubic centimeter. But

$$\frac{dx}{dt} = v,$$

so

$$dn = -\alpha n^2 dt$$

becomes

$$dn = -\frac{\alpha}{v} n^2 dx.$$

Integrating

$$\int_{N_0}^{n} \frac{dn}{n^2} = -\frac{\alpha}{v} \int_0^x dx,$$

$$\frac{1}{N_0} - \frac{1}{n} = -\frac{\alpha}{v} x,$$

from which

$$n = \frac{1}{\frac{1}{N_0} + \frac{\alpha}{v} x}. \tag{2}$$

Case (1) then gives

$$I \propto \frac{1}{(A + Bx)}. \tag{3}$$

Case (2) then gives

$$I \propto \frac{1}{(A + Bx)^2}. \tag{4}$$

In the development of these equations we have assumed that the velocity of the vapor is uniform across a cross-section, while we have good reason to believe that the flow of the vapor is governed by the equations of steady viscous flow. Nevertheless the above equations will hold regarding the luminosity along a fine elementary tube of infinitesimal cross-sectional area with its axis along a streamline of the actual tube.

Let us now consider the vapor as flowing along the tube as governed by the equations of steady viscous flow. Let us also suppose that the velocity is independent of x, and that at $x = 0$ the number of positive ions per cubic centimeter is uniform across the cross-section of the tube and is equal to N_0. Then, if U is the velocity of the vapor along the axis of the tube, the velocity v at a radius r from the axis will be given by

$$v = U\frac{a^2 - r^2}{a^2}, \tag{5}$$

where a is the radius of the tube. The number of ions per cubic centimeter now becomes a function of both x and r.

$$n = \frac{1}{\frac{1}{N_0} + \frac{\alpha a^2}{U(a^2 - r^2)}x}, \tag{6}$$

which may be written as

$$n = \frac{A}{C - Dr^2} - \frac{Br^2}{C - Dr^2}, \tag{7}$$

where

$$A = UN_0a^2, \tag{8}$$

$$B = UN_0, \tag{9}$$

$$C = Ua^2 + N_0\alpha a^2x, \tag{10}$$

$$D = U. \tag{11}$$

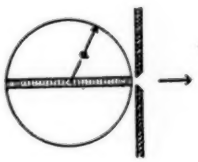


Fig. 11.

Now in making observations on the intensity of the luminosity in the tube, the light which is observed is the light from all points on a diameter of the tube. The total light observed may then be written in the form of an integral.

$$I = 2\int_0^a I_r dr, \tag{12}$$

where I_r is the intensity of the light from a point at a radial distance r from the axis of the tube. In the above, x is held constant. Let us now

take case 1, where I_r is proportional to n. Except for a constant we may then write

$$I_1 = \int_0^a n \cdot dr$$

the subscript 1, referring to case 1.

$$I_1 = A \int_0^a \frac{dr}{C - Dr^2} - B \int_0^a \frac{r^2 dr}{C - Dr^2}. \tag{13}$$

We may, at any time, drop any constant factor, since the unit in which I is measured is arbitrary.

$$I_1 = \frac{A}{D} \int \frac{dr}{\frac{C}{D} - r^2} - \frac{B}{D} \int \frac{r^2 dr}{\frac{C}{D} - r^2}.$$

Let $C/D = b^2$ and for r substitute $b \cos \beta$ then

$$dr = - b \sin \beta \cdot d\beta.$$

We then have

$$I_1 = \frac{- A}{Db} \int_{r=0}^{r=a} \frac{d\beta}{\sin \beta} + \frac{Bb}{D} \int_{r=0}^{r=a} \frac{\cos^2 \beta}{\sin \beta} \cdot d\beta.$$

Upon integrating and dividing out by constant multipliers we have

$$I_1 = 1 + \frac{b^2 - a^2}{2ab} \log \frac{b - a}{b + a}.$$

Substituting the value of b and calling $N_0\alpha/U = k$ this becomes

$$I_1 = 1 - \frac{kx}{2(1 + kx)^{1/2}} \log \frac{\sqrt{1 + kx} + 1}{\sqrt{1 + kx} - 1}. \tag{14}$$

Let us now take case 2, where I_r in (12) is proportional to n^2. Except for a constant factor we may then write

$$I_2 = \int_0^a n^2 dr,$$

which by (7) becomes

$$I_2 = A^2 \int_0^a \frac{dr}{(C - Dr^2)^2} - 2AB \int_0^a \frac{r^2 dr}{(C - Dr^2)^2} + B^2 \frac{r^4 dr}{(C - Dr^2)^2}. \tag{15}$$

If we let $C/D = b^2$ and $r = b \cos \beta$ and integrate we obtain

$$I_2 = 1 + \frac{kx}{1 + kx} - \frac{1 + \frac{3}{4}kx}{(1 + kx)^{3/2}} kx \log \frac{\sqrt{1 + kx} + 1}{\sqrt{1 + kx} - 1}, \tag{16}$$

where $k = N_0\alpha/U$.

XIV. EXPERIMENTAL VERIFICATION OF THE THEORY.

In order to measure experimentally the rate of decay in luminous intensity with the distance along the tube, the apparatus shown in Fig. 12 was made. This apparatus consisted of a glass tube 2.5 centimeters in diameter in which the arc was formed. The vapor from this arc distills upward through the anode and into a tube 3.7 centimeters inside diameter and 65 centimeters long. In the lower part of this tube is an auxiliary grid, A, for use in measuring the velocity of the stream. A long cylindrical electric furnace was placed about the greater part of this tube to prevent the mercury vapor condensing on the walls. There was a long slit along the side of the furnace, parallel with its axis, and about one centimeter wide, through which the luminosity was observed. The arc was "struck" by vaporizing the mercury at the cathode by means of the cathode heating coil, and starting the arc by means of an induction coil between the electrode A and the anode of the arc. The arc operated well with a current of three amperes with the cathode heater going. The potential drop across the arc varied from 13 volts to 20 volts, depending upon the rate of volatilization of the mercury at the cathode. The potential drop was generally about 15 volts.

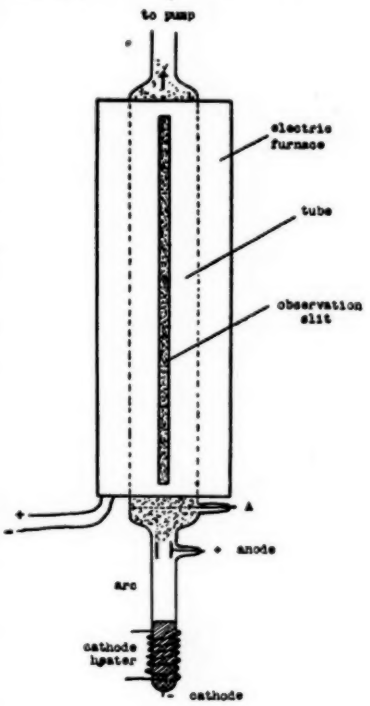


Fig. 12.

The luminous vapor could be traced for the entire length of the tube. The inside walls of the furnace were carefully blackened, and in this way reflected light from the walls was almost completely done away with. In fact when very little vapor was distilling, the upper part of the tube appeared quite dark. The mercury vapor condensed on the cold parts of the tube well above the furnace. The condensed drops of mercury rain down and fall back to the cathode. However the drops have to be fairly large before they can do this. If the extreme upper part of the tube is carefully observed, the small droplets may be seen to fall into the tube. However, before they have descended more than about a centimeter, the uprushing current of mercury vapor strikes them and sends them up again. Thus the mercury must descend in a few large drops

instead of a large number of small ones. This is a good thing, for if there were a large number of small droplets continually raining down into the tube, the light from the arc, which they would reflect, would interfere with the observed luminosity of the distilled vapor. An occasional large drop does not matter. For a given mass of mercury, the area of the reflecting surface is much smaller when the mercury is in one large drop than when it is divided up into a large number of small ones.

The luminosity as a function of the distance along the tube was measured in the following manner. A Nutting polarization photometer made by Adam Hilger was turned on its side and mounted on the carriage of a cathetometer. Aluminium tubes 10 centimeters long were affixed to the apertures of the photometer as shown in Fig. 13. These tubes were carefully blackened on the inside with soot in order to minimize the effect of scattered light entering the photometer. The apertures on the photometer were 3.8 centimeters apart. The device was so arranged that the light from the lower aperture could be decreased by the crossed nicols and so matched with the light from the upper aperture. In order to measure the intensity of the luminosity along the vapor stream the ratio of intensities between these two points was observed. The photometer was then moved up so that the lower tube came into the position previously occupied by the upper tube. The ratio was again determined, and so on for about two thirds of the length of the tube. Ten such ratios were observed along the tube at intervals of 3.8 centimeters. The ends of the tube were avoided as it was feared that conditions might not be steady and uniform near the ends. The distance between the photometer tubes, 3.8 centimeters, was taken as unit distance, and at $x = 0$ the intensity of the light was taken as unity. So the intensity at any point could be calculated as the product of all previous ratios. Several series of readings were made of the luminosity as a whole. These readings agreed very well with one another. Other readings were made with the mercury green line alone by holding a mercury green line filter in front of the eyepiece. The decay of the mercury green line gave a curve almost identical with the decay of the luminosity as a whole. In Fig. 14, I_x shows the manner in which the luminosity decays as the vapor flows along the tube.

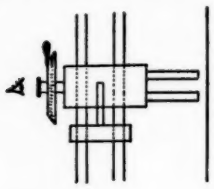


Fig. 13.

Now to compare these experimental curves with the theory of the previous chapter. It will be seen in the expressions for I_1 and I_2 that their graphs will both pass through the point $x = 0$, $I = 1$, and that the luminosity vanishes as x becomes infinite. There is, however one

arbitrary constant in each of the expressions (14) and (16). This means that, by properly choosing the constants, we can make the curves pass through one arbitrary point on the experimental curve. If in (14) we take $k = 1.6$ and in (16) we take $k = .36$ the graphs of these equations will both pass through the point $x = 5$, $I = .076$ approximately. These curves are also shown in Fig. 14. It will be seen that neither of these two theoretical curves fit the experimental curve very well. However it will be noticed that the curve I_2 is not so far away from the experimental curve as is I_1. That is to say, if we are to decide between recombination producing the light, and the first case considered in the previous chapter, the evidence is decidedly in favor of the recombination theory.

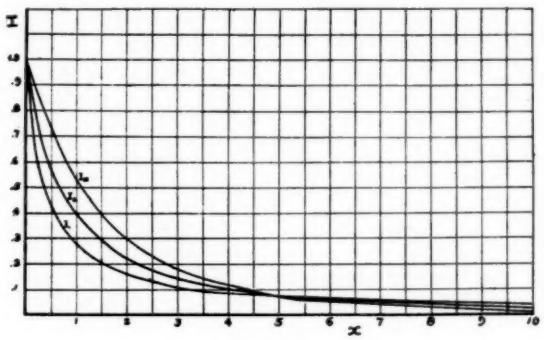


Fig. 14.

In the previous chapter it was stated that the third case would be considered later. This case assumes that the positive ions emit the light continually during their journey from the arc. If the light from each ion decays exponentially with respect to the time we should expect the luminosity at a point to be proportional to $ne^{-\lambda t}$. So

$$I \propto \frac{e^{-\lambda x}}{1 + Ax}.$$

Equations of this form will involve two arbitrary constants, and, after an integration procedure as in cases 1 and 2 has been carried out, the two arbitrary constants will enable us to pass the curve through two arbitrary points of the experimental curve. Thus this theory might be made to agree with the experimental curve. However it must be borne in mind that as soon as a number of arbitrary constants enter we can make an equation fit almost any sort of curve. Thus the equation becomes little more than an empirical one. We cannot then give as much weight to such a curve fitting the experimental curve as when but a single arbitrary constant is involved.

In matching intensities with the Nutting photometer it was found that there was a color difference in the lights entering the two apertures. The light entering the lower aperture appeared to be yellower than that entering the upper aperture. Thereupon the following experiment was tried. The photometer was kept in a fixed position and the individual lines matched separately. This was done by holding filters before the eyepiece so that only the light under observation was transmitted. In this way the decay in the intensities of the separate lines over the distance between the photometer tubes could be measured. These ratios were measured for the yellow lines, the green line, and the violet line—$\lambda = 4{,}358$. The average of a large number of observations showed that, for equal intensities starting at the first aperture, for each unit of the violet luminosity reaching the second aperture, .873 units of the green luminosity, and .786 units of the yellow luminosity reach the second aperture. This shows that the violet line persists longer than the green line and that the green line persists longer than the yellow lines.

XV. CONCLUSION.

In the preceding chapters it has been shown that the observed phenomena with the luminosity and electrical conductivity of the distilled vapor can be explained on the assumption that the luminosity is produced during the recombination of positive and negative ions to form neutral molecules, and that the intensity of the luminosity is accordingly proportional to the rate of recombination of the ions. Some of the phenomena can be explained by the positive ions themselves emitting the light; and with one phenomenon (positive effect) it does not seem possible to explain the effects except on the recombination theory. On the whole the evidence is very much in favor of the recombination theory.

In conclusion the author wishes to express his most sincere thanks to Professor H. A. Wilson for his kindly interest in these experiments, and for his helpful advice during the course of the investigation.

PHYSICS LABORATORIES,
THE RICE INSTITUTE,
HOUSTON, TEXAS.

HENRY ANDREWS BUMSTEAD

HENRY ANDREWS BUMSTEAD, Professor of Physics and Director of the Sloane Physical Laboratory at Yale University, Chairman of the National Research Council, Past-president of the American Physical Society and a former member of the editorial board of the PHYSICAL REVIEW, died on the train on the thirty first day of December, 1920, while returning from the Chicago meeting of the Physical Society and the American Association for the Advancement of Science. Professor Bumstead was in his fifty first year.

PROCEEDINGS

OF THE

AMERICAN PHYSICAL SOCIETY.

MINUTES OF THE CLEVELAND MEETING, NOVEMBER 26 AND 27, 1920.

THE 105th regular meeting of the American Physical Society was held in Cleveland, Ohio, on November 26 and 27, 1920. The morning session of November 26 was held in the Physical Laboratory of Western Reserve University and the sessions in the afternoon of November 26 and in the morning of November 27 were held in the Physical Laboratory of Case School of Applied Science. The presiding officers were President Ames and Vice-President Lyman. There were about 125 members and friends in attendance. There was a dinner at the University Club on the evening of November 26, attended by about sixty-five persons; an enjoyable feature of the dinner being the attendance of Dr. T. C. Mendenhall. Complimentary luncheons, given by Western Reserve University and Case School of Applied Science, were served on the days of meetings. The college laboratories were inspected by many members during the intermissions, and after adjournment on November 27 many members accepted the invitation to inspect the Nela Research Laboratories. The meeting was considered one of the most enjoyable of the smaller meetings.

Two Committees of the National Research Council took advantage of the occasion to hold meetings of their respective committees, the Committee on Acoustics and the Committee on Methods and Technique of Physical Research.

At the meeting of the Council on November 26, attended by six members, the following business was transacted:

The proposal of the Board of Editors to issue a general index to the PHYSICAL REVIEW covering the issues from 1893 to 1920 inclusive was approved.

Nomination of officers and preparation for the program of the Annual Meeting were made according to other announcements of the Secretary.

One hundred and sixteen persons were elected to *Associate Membership*, as follows: Paul Anderson, Jeanette Armstrong, Edward J. Baldes, A. F. Beal, Arthur L. Becker, George Becker, A. E. Bellis, Donald M. Bennett, S. L. Boothroyd, Frederick S. Brackett, Ben E. Brown, W. Byron Brown, Charles Brush, Jr., A. B. Bryan, Jay W. Buchta, Johanna Busse, Lois Carr, R. A. Castleman, Jr., Elliott W. Cheney, Paul P. Cioffi, J. R. Collins, Jean P. Cooley, Howard L. Coyne, J. W. Doolittle, Philip H. Dowling, Katherine M. Downey,

James A. Duncan, John B. Esmaker, Louisa L. Eyre, Joseph L. Finkelstein, Peter E. Fossum, Willena Foster, James B. Friauf, Hugo Fricke, Paul H. Geiger, L. H. Germer, Victor Guillemin, Rowland V. Hagen, Lloyd B. Ham, A. Baird Hastings, Lucy J. Haynes, George C. Henny, Erik Hetle, W. J. Hitchcock, Jean Huddleston, C. Judson Humphreys, Charles W. Jarvis, Edward T. Johnson, A. L. Johnsrud, E. W. Kellogg, K. H. Kingdon, Elias Klein, E. J. Knapp, Chitting Kwei, C. E. Lane, Earl E. Libman, Ernest F. Ling, Laurice L. Lockrow, R. E. Lofton, William V. Lovell, Erwin F. Lowry, William A. Lynch, Hector J. MacLeod, Leopold D. Mannes, John Richard Martin, R. Earl Martin, Frederick A. May, Wm. Walter Merrymon, Melvin Mooney, Charles R. Moore, Clement Moran, Harold M. Mott-Smith, Waldemar Noll, Clarence A. Nordquist, Russell M. Otis, Samuel R. Parsons, George M. Pearsall, James O. Perrine, C. J. Pietenpol, H. H. Plaskett, George A. Pomeroy, James R. Randolph, Leo G. Raub, F. W. Reynolds, Martin E. Rice, Harold F. Richards, Donald E. Richmond, Paul Rood, P. M. Roope, Wilfred N. St. Peter, Otto Sandirk, Ralph A. Sawyer, Merit Scott, Marcus L. Sherin, Roger H. Sinden, John C. Slater, T. S. Sligh, Russell P. Smith, Sinclair Smith, Theodore A. Smits, B. L. Steele, I. Melville Stein, Homer G. Tasker, A. Hadley Taylor, Lucien B. Taylor, Harry C. Thompson, Jesse L. Van Horn, Robert O. Von Nardroff, John H. Van Vleck, Albert S. Walton, John S. Ward, Yu Ching Wen, R. B. Wilsey, Evelyn Wise, E. W. Woolard, Franklin B. Wright.

The following program of thirty-one papers was presented, five being read by title:

The Crystalline Structures of Sputtered Metallic Films. H. KAHLER.

The Effect of Adsorbed Air on the Aging of Thin Metal Films. LEWIS R. KOLLER.

Constancy of Hall Coefficient in Thin Silver Films. G. W. STEWART.

The Hall Effect and the Specific Resistance in Silver Films. G. R. WAIT.

The Hall Effect of the Nernst Effect in Magnetic Alloys. ALPHEUS W. SMITH.

Conductivity of Insulating Materials Near the Breakdown Voltage. J. E. SHRADER.

Magneto-Resistance Effects in Films of Bismuth. L. F. CURTISS.

The Magnetic Mechanical Analysis of Manganese Steel. SIR ROBERT HADFIELD, F.R.S. (London), and S. R. WILLIAMS and I. S. BOWEN.

Pulsating and Thermionic Discharges in Evacuated Tungsten Lamps. A. G. WORTHING.

Effect of Strong Electrostatic Fields on the Vaporization of Tungsten. A. G. WORTHING, and W. C. BAKER.

A Compensation Scheme for Electrostatic Measurements. W. F. G. SWANN. (Read by title.)

Note on Electromagnetic Induction and Relative Motion—A Rejoinder. W. F. G. SWANN. (Read by title.)

The Electrostatic Charges of the Earth and Sun and Their Relation to Terrestrial Magnetism. FERNANDO SANFORD.

The Spectro-Photoelectric Sensitivity of Proustite. W. W. COBLENTZ. (Read by title.)

Color Filters for Photographic Uses. CHARLES D. HODGMAN.

Comparison Between the Fundamental Equations for Ponderomotive Force for Point Charges Due to Larmor-Lorentz and to Mega Nad Saha. ALBERT C. CREHORE.

The Newtonian Laws of Gravitation Deduced from the Saha Electromagnetic Theory Applied to the Copernican Atom. ALBERT C. CREHORE.

On the Propagation of the Sound Wave from the Muzzle of a Large Gun. DAYTON C. MILLER.

A New Tone Generator. C. W. HEWLETT.

A Method of Studying Sound Waves by Means of a Synchronous Commutator. L. O. GRONDAHL.

On the Relative Positions of Lines in X-Ray Spectra. WILLIAM DUANE and R. A. PATTERSON.

X-Ray Analysis of Three Series of Alloys. MARY R. ANDREWS.

The Relation Between the Emissive Power of a Metal and Its Electrical Resistivity. C. DAVISSON and J. R. WEEKS.

Mass-Absorption Coefficients as a Function of Wave Length Above and Below the K X-Ray Limit of the Absorber. F. K. RICHTMYER.

Graphical Determination of Hexagonal and Tetragonal Crystal Structure from X-Ray Data. ALBERT W. HULL and WHEELER P. DAVEY.

The Absorption and Scattering Coefficients for Homogeneous X-Rays in Several Elements of Low Atomic Weight. C. W. HEWLETT.

The Current-Temperature Relation for Different Pyrometer Filaments. W. E. FORSYTHE.

On the Absorption Spectrum of HCl. WALTER F. COLBY and CHALES F. MEYER.

Flame Excitation of Luminescence. E. L. NICHOLS and D. T. WILBER.

A Continuous Spectrum from Mercury Vapor. C. D. CHILD. (Read by title.)

Note on Sheet Striae as Thin Films in Optical Glass. L. E. DODD. (Read by title.)

The abstracts of all the papers on the above program are given in the following pages.

DAYTON C. MILLER,
Secretary.

THE CRYSTALLINE STRUCTURE OF SPUTTERED METALLIC FILMS.

BY H. KAHLER.

USING the X-ray method of crystal analysis cathodically sputtered films[1] of bismuth, silver, gold, cobalt and platinum, have been examined. *All have been found to be crystalline with a random orientation of crystal axes.*

[1] Prepared by L. F. Curtiss.

The lattice structures and dimensions of these films have been determined and in every case have been found to be identical with those of the unsputtered metals. The cases of cobalt, and selenium present complications due to their allotropic forms. The nature of these complications is now being studied.

A comparison is also being made between the crystalline structures of sputtered and evaporated films of silver.

CORNELL UNIVERSITY,
November, 1920.

THE EFFECT OF ADSORBED AIR ON THE AGEING OF THIN METAL FILMS.

BY LEWIS R. KOLLER.

THIN metal films deposited by cathode sputtering in vacuo undergo a decrease in resistance with time. This process has been called ageing. In the course of some senior research on ageing in this laboratory it was noted that the films apparently increased in resistance when air was admitted to the sputtering jar and again decreased when the jar was reëxhausted.

In this investigation thin platinum films were prepared by sputtering in air, and the ageing was studied first in the best obtainable vacuum and then in air at atmospheric pressure. In the vacuum the resistance of the film at first decreases very rapidly and then more slowly, finally reaching a nearly constant value. The curve of resistance plotted against time is regular in form and values can be closely duplicated for different films. The effect of

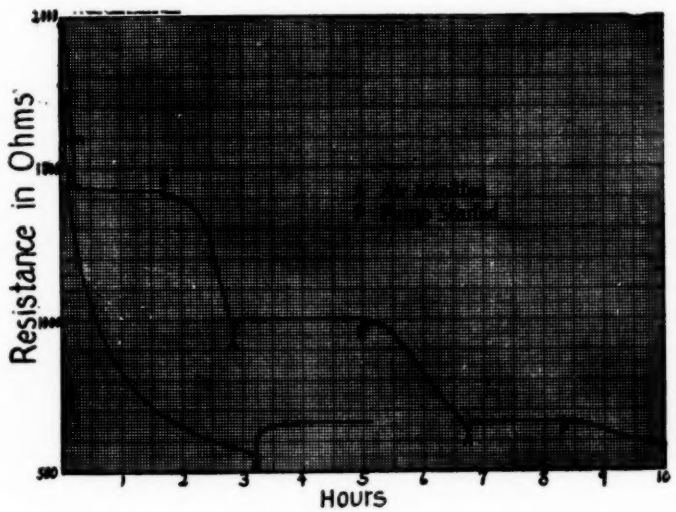


Fig. 1.

admitting air at any time during the process is practically to stop the ageing. When the air is admitted the resistance of the film shows a slight initial rise (along a definite curve) after which it remains practically constant.

If a sputtered film is considered as composed of many small particles the

initial aging in vacuo is probably due to the agglomeration of these particles. The effect of admitting air is to surround each of these particles with a film of adsorbed gas, thus preventing further agglomeration. If a film which has been allowed to adsorb air is again placed in a vacuum, the adsorbed air will be given off and the film will continue to age as before. The ageing can again be halted at any point by admitting air and can be continued again by pumping out the jar. The accompanying curves show this process of ageing in stages and normal ageing in vacuo. This effect in particular is a strong indication that adsorbed gases play a large part in determining the properties of thin metal films.

When the cover of the sputtering jar is removed and the film is allowed to come into contact with the air of the room its resistance undergoes a sudden decrease. It is not likely that this is due to the condensation of a water film as the decrease is very much greater than would be caused by a water film. This effect is at present under investigation.

The writer is pleased to acknowledge assistance from a grant from the Rumford Fund to Prof. F. K. Richtmyer.

PHYSICAL LABORATORY OF CORNELL UNIVERSITY,
November, 1920.

CONSTANCY OF HALL COEFFICIENT IN THIN SILVER FILMS.

BY G. W. STEWART.

IN the preceding abstract Dr. Wait announces that the Hall coefficient retains the same value as in bulk silver even though the film used may have a specific resistance many hundred times that of bulk silver. The constancy of the coefficient suggests a theoretical examination in order to ascertain the simplest set of assumptions that will account for the result and yet without conclusions not in accord with any results obtained upon the variation of specific resistance of thin films.

If the film be supposed to consist of granules, each treated as bulk silver, and the current to be conducted from one granule to the next through areas of "contact" across which the electrons pass, then the following assumptions seem reasonable:

1. That, in any normal cross section of the film, the current perpendicular to the cross section flowing through granules is large in comparison with the current flowing in the contact gaps.

2. That the total fall of potential across the gaps between the granules in the direction of the Hall E.M.F. is small compared with this E.M.F.

With these two assumptions it is readily found that the Hall coefficient is independent of the thickness of the films. There is no qualitative disagreement of the theory with experiments upon specific resistance; it is therefore important to pursue experiments suggested by the theory. Films should be obtained having much smaller granules, so small that the contact gaps are comparable in size, and the constancy of the coefficient tested. Tests for

constancy should be made in films of metal having a coefficient of opposite sign.

PHYSICAL LABORATORY,
STATE UNIVERSITY OF IOWA.

THE HALL EFFECT AND THE SPECIFIC RESISTANCE IN SILVER FILMS.

BY G. R. WAIT.

THE purpose of the investigation was to establish with greater certainty the variation of the Hall effect in thin silver films and simultaneously to compare the variation of specific resistance. The work was limited to chemically deposited films that were hard and, in contrast to previous experiments, mechanically permanent. The thickness as determined by weighing the silver and also by weighing the silver iodide resulting therefrom, varied from about 20 $\mu\mu$ to 200 $\mu\mu$. The chief results were as follows:

1. The specific resistance rapidly increased with decreasing thickness at 24 $\mu\mu$ to 30 $\mu\mu$. This result is similar to those of previous investigations.

2. There was found a constant Hall coefficient; *i.e.*, the Hall effect was proportional to the primary current and to the magnetic field and inversely proportional to the thickness of the films. Moreover, this coefficient proved to be, within errors of experiment, the same as that found in bulk silver. This result is new.

3. Incidentally but importantly a modified process of deposition was found which produced films that were hard, that were reproducible and that remained unchanged with time, chemical action excluded.

It can be shown that the assumption of conduction between adjoining granules makes possible the explanation of the constancy of the Hall coefficient. Moreover with this assumption we can explain qualitatively the change of specific resistance with thickness and with temperature.

PHYSICAL LABORATORY,
STATE UNIVERSITY OF IOWA.

THE HALL EFFECT AND THE NERNST EFFECT IN MAGNETIC ALLOYS.

BY ALPHEUS W. SMITH.

The Hall Effect.—Observations have been made on the Hall effect in iron-copper, nickel-copper and iron-nickel alloys. The addition of small quantities of copper to iron or nickel increases the Hall effect. In the iron-copper series the Hall effect has its largest value in an alloy containing 1.5 per cent. copper. A further increase in the amount of copper causes a decrease in the Hall effect. The variation of the Hall constant in this case is very similar to the variation of the electrical resistance under corresponding conditions. In the nickel-copper series the addition of copper continues to increase the Hall effect until the alloy contains a little more than 26 per cent. of copper. This alloy corresponds to the compound $CuNi_3$. When the concentration of copper is increased

beyond that necessary for this compound, the Hall effect drops suddenly to a small fraction of its largest value, and then decreases gradually with a further increase in the concentration of copper. The Hall effect is positive in iron and negative in nickel. Nevertheless the addition of nickel to iron causes a rapid increase in the Hall effect so that an alloy containing 13.11 per cent. of nickel shows a Hall effect which is six times its value in pure iron. There is an approximate proportionality between the Hall effect and the concentration of nickel in the alloy for alloys containing not more than 13.11 per cent. of nickel.

The Nernst Effect.—The curve showing the relation between the Nernst effect and the concentration of copper in a nickel-copper series of alloys has nearly the same form as the corresponding curve for the Hall effect. Each curve shows a break where the concentration is that necessary for the compound $CuNi_3$. The Nernst effect is negative in iron and positive in nickel. When nickel is added to iron the Nernst effect decreases in magnitude, becomes zero for a concentration of about 2.2 per cent. nickel and then reverses its direction for higher concentrations of nickel. When the alloy contains 13.11 per cent. of nickel the effect has the direction which it has in pure nickel and its magnitude is about five times as large as in pure iron.

Reversal of the Hall Effect in Alloys.—An attempt is then made to find in the theory of Borelius an explanation of the reversal of the Hall effect in certain alloys in which the effect is negative for small values of the magnetic field and positive for large values. The observed effect can be broken up into two parts, a positive part which is proportional to the magnetic field and a negative part which at first increases with the magnetic field and then reaches a limiting value. The theory requires that this part of the Hall electromotive force be proportional to the magnetic field and this is in agreement with the observations. By assuming that the negative part required by the theory reaches a limiting value when the magnetic field is sufficiently increased, satisfactory agreement between theory and observation is again obtained.

Conclusion.—The direction and magnitude of the Hall effect and the Nernst effect must be thought of as determined by the crystal lattice and the fields of force in the intermolecular spaces rather than by the characteristics of the metal atoms or the deflection of free electrons in the interstices between the atoms.

OHIO STATE UNIVERSITY.

CONDUCTIVITY OF INSULATING MATERIALS NEAR THE BREAKDOWN VOLTAGE.

BY JAMES E. SHRADER.

A METHOD of testing insulation whereby the values of conducting are obtained just before breakdown has been devised which provides against danger to the recording instruments in case of puncture.

This method consists in placing an insulated hot cathode rectifier in series with the sample which is being tested by voltage from a D.C. source or rectified A.C. By controlling the temperature of the cathode of the rectifier used as a

current limiting device the value of the current flowing is always under the control of the operator. When under a given voltage the current does not reach a constant value upon raising the temperature of the cathode, this is an indication that the voltage has reached the breakdown point. Actual breakdown can occur by this device without damage to the recording instruments.

With this device samples of insulating material have been tested up to the point of breakdown. From inspection of current-voltage curves it is observed that these materials fall into three general classes.

1. Materials whose conductivity varies directly with voltage according to Ohm's Law.

2. Materials whose conductivity varies according to Ohm's Law up to the neighborhood of breakdown after which the conductivity increases at an accelerated rate to breakdown.

3. Materials whose conductivity increases over the whole range at an accelerated rate to breakdown.

WESTINGHOUSE RESEARCH LABORATORY,
November 10, 1920.

MAGNETO-RESISTANCE EFFECTS IN FILMS OF BISMUTH.

BY L. F. CURTISS.

THE work on films of bismuth obtained by cathodic sputtering, of which a report[1] has previously been made, has been continued with some interesting results.

Films prepared in essentially the same way as those used previously were mounted in flat glass tubes with leads sealed through the glass. These tubes were evacuated for several hours at a temperature of 200°–230° with a Langmuir mercury vapor pump. They were then sealed off and the films thus mounted were heated in an oil bath for three or four hours at about the same temperature and at intervals several days apart. The purpose in this was to complete the ageing and to bring about the steady conditions which the previous work had indicated as possible. After this result had been attained the films were placed in a temperature bath between the poles of the magnet and measurements made of the change of resistance at various field strengths and at several constant temperatures from 20° to 230°.

This procedure yielded several new results. First, and perhaps most striking, was the fact that the films throughout this process of heating as described above steadily *decreased* in resistance down to a final value after the heating had been continued long enough. After this state had been reached accurately reproducible results in the measurement of resistance at various temperatures could be obtained provided the film was not raised to a higher temperature than the previous maximum. Both these facts are at variance with previous experience with films of this metal for, as will be recalled, all former films (with

[1] F. K. Richtmyer and L. F. Curtiss, PHYS. REV., XV., p. 465, 1920.

one or two exceptions) increased in resistance on standing after preparation and this rate of increase was hastened by heating. Furthermore there seemed to be no limit short of infinity for the value of the resistance of a film after successive repetitions of the heating, so that reproducible measurements were never attained. The explanation for this difference seems obviously to lie in the difference in the manner of protecting the films in the two cases. Formerly the films were heated in oil which undoubtedly contained some air so that the films were oxidized whereas in the present case the air was pretty well removed from the tube in which the film was placed and thus oxidation was prevented to a greater extent. In addition the oil taken up by the film in the former case and the removal of a certain amount of adsorbed air from the film in the second case contributed also to this difference without doubt. Thus the present results indicate that bismuth films behave during ageing in the same way as films of platinum, gold, and silver have been found to behave by other investigators, at least as far as the decrease of the resistance of the film is concerned.

It was found further, in the measurement of the resistance of the films at various temperatures, that the negative temperature coefficient became smaller in value as the temperature increased, finally becoming zero and reversing its sign, and attaining a positive value which gradually increased in magnitude as the temperature was further increased. Thus the temperature-resistance curve has a minimum, the location of which depends on the initial resistance of the film, being in general at a lower temperature the less the resistance of the film, *i.e.*, the thicker the film, since all the films had the same length and breadth. This curve is, moreover, perfectly symmetrical with respect to an axis parallel to the y-axis (plotting resistance on the y-axis and temperature on the x-axis). The greater the resistance of a film the larger the value initially of the negative temperature coefficient.

Films treated in this manner also showed a much greater increase of resistance in the magnetic field than did any of the films in the former work. The maximum obtained was about 17½ per cent. at a field of about 16,000 gauss and at a temperature of about 20°. For the previous work the corresponding maximum was approximately 4 per cent. These measurements were also reproducible provided the heating had been carried far enough so that the temperature-resistance curve had attained its final form. The increase of resistance is not proportional to the square of the field strength, as it seemed to be before. When the percentage increase of resistance (y-axis) is plotted against the square of the field (x-axis) the observed points outline a curve concave towards the x-axis, resembling very closely the corresponding curve for bismuth in bulk, except for the fact that the effect has only about one quarter the magnitude. When the percentage increase of resistance is plotted directly against the field the curve bends upward quite perceptibly at first but finally takes up a definite direction so that the greater part is a straight line, again agreeing with the behavior of bismuth in bulk. This similarity is further shown in the results obtained at various temperatures. The effect

becomes less as the temperature is increased above room temperature, at first dropping rapidly, so that if the percentage increase of resistance (y-axis) is plotted against the temperature (x-axis) curves of gradually decreasing curvature are obtained, convex towards the x-axis, hence very similar in form to those for bismuth in bulk under the same conditions.

CORNELL UNIVERSITY,
November 1, 1920.

THE MAGNETIC MECHANICAL ANALYSIS OF MANGANESE STEEL.

BY SIR ROBERT HADFIELD, F.R.S., S. R. WILLIAMS AND I. S. BOWEN.

BY proper heat treatment, so-called non-magnetic manganese steel may be made magnetic. From a group of six manganese steel rods, drawn from the same source, three were heat treated and three untreated. Rods, numbered 5 and 2 were selected as representatives of these two classes respectively.

These rods were used in a series of comparative tests between the reciprocal magnetic and mechanical properties, known as the Joule effect (change in length due to a magnetic field) and the Villari effect (change in magnetic intensity due to a longitudinal stress).

The heat treated rod showed an increase in length for all field strengths employed, consequently there was no Villari reversal but an increase in magnetic intensity for these same field strengths when the rod was stretched.

The so-called non-magnetic sample showed some magnetic intensity when subjected to a magentic field. If we multiply the values of I for the non-magnettc rod by the factor 36 we practically duplicate the values of I for the magnetic rod. It would indicate that the heat treatment produced an added quantity of something which was very meagerly supplied to the non-magnetic rod.

The remainder of the paper deals with the importance of magnetic mechanical analysis, (1) from the standpoint of applied science, (2) for its bearing on a comprehensive magnetic and atomic theory.

LONDON, ENGLAND,
OBERLIN, OHIO.

PULSATING THERMIONIC DISCHARGES IN EVACUATED TUNGSTEN LAMPS.

BY A. G. WORTHING.

THE starting point of this investigation has been the observation of peculiar, periodic, automatic changes in brightness of a certain tungsten ribbon filament in an evacuated lamp bulb. Several other lamps were investigated, only a few of which showed similar discharges, and none to the extent shown by the one in which the discovery was made.

All of the lamps tested contained a third insulated terminal by means of which thermionic discharges in the lamp could be measured. Simultaneous

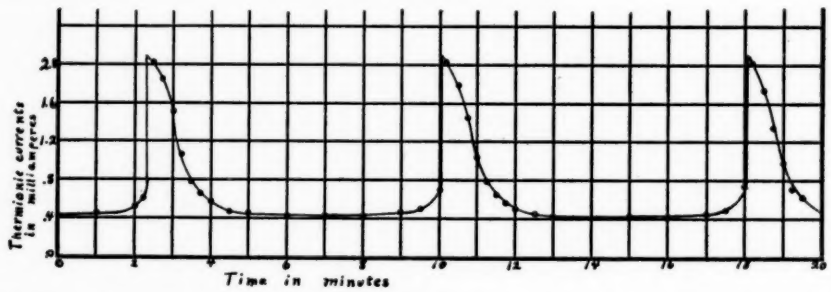
observations of thermionic currents and of filament brightnesses by means of an optical pyrometer showed conclusively that the original variations noted in the filament brightness were directly connected with changes in the thermionic discharges in the lamp. These discharges may take place either from the filament to the insulated terminal or from one leg of the filament to the other.

The tungsten ribbon filament in the lamp tested was V-shaped with sides about 2.5 cm. long and with an apex slightly truncated so as to give an approximately square surface 1 mm. on edge which in the mounting faced the third terminal also of tungsten. The lamp was evacuated in the ordinary manner.

In one test, a direct current of 8.2 amperes was used in heating the filament to a temperature of about 2280° K. The negative terminal of a 44-volt battery was connected to one of the leads to the ribbon filament through a voltmeter with 5,000 ohms resistance; the positive terminal of the battery was connected to the insulated terminal in the bulb.

The thermionic currents represented by the voltmeter readings showed discharges of a pulsating character (see accompanying figure) in which, without any alteration of conditions by the experimenter, the currents rose from a moderately steady minimum at first slowly and then with great suddenness to a maximum value at least 4.8 times the minimum value, and then fell more or less gradually to the original minimum value. The period of the discharge, eight minutes, was constant for nine successive intervals to within 4 per cent. Small characteristic variations in the form of the curves were repeated also.

Different temperatures of operation, different applied voltages and resistances in the voltmeter circuit yielded discharges differing in period, minimum and maximum current values, and ratios of maximum to minimum current



values. Temperatures beyond a certain value and resistances below a certain minimum in the thermioinc current circuit, other conditions remaining unchanged, tended toward pulsations which gradually diminished in intensity and disappeared leaving a steady thermionic discharge.

No definite explanation is offered for this action. However, the blue glow discharges which were obtainable in the lamps used would seem to indicate that it might be connected with the residual gases in the lamps.

NELA RESEARCH LABORATORIES,
CLEVELAND, OHIO,
November, 1920.

Effect of Strong Electrostatic Fields on the Vaporization of Tungsten.

By A. G. Worthing and W. C. Baker.

THE general procedure has consisted in noting the rate of change of resistance of a tungsten filament operated in vacuo at some fixed current first when exposed to a weak electrostatic field opposing thermionic emission and second when exposed to a similarly directed strong electrostatic field. For well-seasoned filaments, the changes in resistance with continued burning, at least for moderate electrostatic fields, are due largely to changes in filament cross section resulting from vaporization. Therefore, any difference in the rates of change of resistance for the two conditions of operation is naturally to be ascribed to the effect of the strong electrostatic field on the rate of vaporization of the material.

The most satisfactory mounting for the filaments studied contained two insulated circuits leading through the stem of a well-evacuated, hard-glass lamp-bulb. One of these circuits included the filament to be investigated held straight and taut by springs of larger tungsten wire, in the axis of a helical coil of still larger tungsten wire which formed a part of the other insulated circuit. All wires leading into the lamp bulb, as well as all wires within, were of tungsten. All junctions were fused. The strong electrostatic field was obtained by connecting the insulated coil to the negative terminal of a small motor-driven Winshurst machine while the hot filament and the frame of the machine were earthed.

In the most satisfactory test conducted, a 0.060 mm. filament 3.5 cm. long, mounted in a 6 mm. helix 3.2 cm. long was operated at 2780° K. alternately for several time intervals of equal length, first with a potential difference between filament and coil of 30 volts and second with an average potential difference of 11,000 volts. This latter value corresponded to an electrostatic field strength of 800,000 volts/cm. at the surface of the filament.

The measured resistances platted as functions of time of burning showed a straight line relationship for each interval of burning. The averages of the slopes, four against three, showed no change to within ½ per cent. due to electrostatic field. Hence, we conclude that, for the conditions specified, the effect of the electrostatic field on the vaporization of tungsten is negligible.

According to generally accepted views a filament connected to earth becomes deficient in negative electrons when exposed to a steady negative electrostatic field. This means that there exists in the filament an excess of positive ions during the continuation of the electrostatic field. These are probably located on the surface and may here be called "surface ions" to distinguish them from the very temporary ions throughout the filament which are the counterparts of the temporary free electrons. A field strength of 800,000 volts/cm., mentioned above, means a "surface-ion" density of 44×10^{10} ions/cm.^{2}, one atom of every 70,000 surface atoms being ionized.

Due to the pull of the electrostatic field these positive "surface ions" might

be expected to be more easily detached, that is to be more readily vaporized than neutral surface atoms. Just what forces are called into play and how they are directed is difficult to say since we have no way as yet of determining whether these "surface ions" sustain the whole or only a part of the pull of the electrostatic field on the filament. Thus (1) the tubes of induction from the coil may terminate only on surface ions; (2) the tubes may terminate indifferently on neutral atoms and surface ions, with a readjustment of the tubes within and between the atoms of the filament; or (3) the tubes may congregate in bundles with relatively a large portion of them terminating on "surface ions," and with the remainder terminating on neutral atoms, but with a selective tendency for atoms in close proximity to "surface ions." On supposition (1) the "surface ions" sustain the total pull due to the electrostatic field; on supposition (2) the pulls sustained by a surface ion and a neutral atom are equal; on supposition (3) the pull sustained by the "surface ions" is intermediate between those sustained in cases (1) and (2).

Supposition (3) seems most probable. Some speculation on the basis of (1), however, is perhaps permissible, since in that case the computations, in addition to entailing only a moderate amount of labor, will likely give some idea of the magnitudes of the actual electrostatic forces to which the "surface ions" are subjected. For a field strength of 800,000 volts/cm., the pull on an ion in a moderately smooth filament surface is computed to vary from about 63×10^{-8} dynes to roughly twice that amount, depending on how far out the individual ion may be from the surface described by the remaining "surface ions." This force is of the magnitude of the force computed from the estimated yield stress for a tungsten wire at incandescence reduced to a single atom (yield stress ÷ cross-sectional surface density of atoms). This approximate coincidence together with the failure to detect an effect of an electrostatic field on vaporization does not necessarily indicate any error in supposition (1) or in the computations made, since, on the "amorphous phase theory" of the binding material between grains the yield stress may depend on the weak amorphous phase while the resistance to vaporization depends largely on the cohesive forces within the crystal. Moreover, vaporization takes place by atoms, whereas the yielding of a stressed wire takes place by layers or "en bloc." On the other hand, the approximate coincidence leads us to hope that, by increasing the field strength, or by a modification of conditions, a positive influence of electrostatic fields on vaporization may be found.

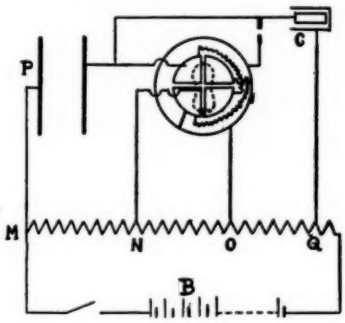
NELA RESEARCH LABORATORIES,
CLEVELAND, OHIO,
November, 1920.

A COMPENSATION SCHEME FOR ELECTROSTATIC MEASUREMENTS.

BY W. F. G. SWANN.

IN the measurement of a saturation current in a gas, it is customary to connect one plate of the ionization chamber to one pole of a high-potential battery, the other plate being connected to one quadrant of an electrometer

whose other quadrant, together with the remaining pole of the battery is connected to the electrometer case. If, under these conditions, the electrometer is to be free from fluctuations resulting from inductive action of the high-potential plate, a very high battery constancy is essential. Again, while a quadrant electrometer is usually adjusted so as to show no deflection on application of the needle potential, with the quadrants connected together, this independence of the position of the needle on the needle potential does not persist when one quadrant is insulated as in actual use, so that here again we are confronted with the necessity of high battery constancy. All difficulties arising from fluctuations in battery potentials may be eliminated by the very simple device indicated in the diagram.



The high-potential battery B is connected to the ends of a high resistance, which may be conveniently made of coils of I.A.I.A. wire immersed in oil. The high potential plate P of the ionization chamber, the electrometer needle, electrometer case, and outer member of a subsidiary condenser C are connected to different points M, N, O, Q, along the resistance. Thus, both the ionization-chamber potential and the needle potential are provided from the same battery. By adjusting the capacity of C, or the position of the point Q, it is readily possible to arrange so that on throwing on the whole battery B there is no inductive deflection of the needle; the various capacities then compensate in their inductive actions, and the whole set-up is absolutely independent of fluctuations in the battery, provided that the resistance ratios remain constant. The relative order of the points M, N, O, Q, may vary with the capacities involved; and, it is even possible, in certain cases, to dispose of C and let the capacity between needle and quadrant function in its stead.

Another useful feature of the above arrangement is its adaptibility in a case where one has a charge between the plates, which it is desired to measure. Ordinarily, it would be necessary to apply the potential to P with the electrometer quadrants connected to the case so as to avoid inductive action. One would then endeavour to release one of the quadrants from earth *as quickly as possible* after applying the potential so as to catch all the ions as they came across. There must always be a certain loss in such an experiment. With the present arrangement, however, the inductive action is compensated,

so that one may release the quadrant from earth before throwing the potential on P, and thus avoid losing any of the charge which it is desired to measure.

By inserting an insulated potentiometer-system, between Q and C for example, it is easy to reduce the measurements to a null method.

The principle of compensation described may of course be extended to such appliances as the tilted-leaf electroscope, etc.

UNIVERSITY OF MINNESOTA.

NOTE ON ELECTROMAGNETIC INDUCTION AND RELATIVE MOTION. A REJOINDER.

BY W. F. G. SWANN.

AT the Washington, D. C., meeting of the American Physical Society, April 23–24, 1920, Dr. S. J. Barnett presented a paper[1] in which he takes exception to a paper by myself presented before the Physical Society on December 30, 1919[2], and dealing with an experiment performed by him on Electromagnetic Induction and Relative Motion.[3] As Dr. Barnett appears to consider that my conclusion is irrelevant, and that I have overlooked certain former papers of his own which, it is claimed, cover the point in question, I may perhaps be permitted to restate what the conclusion really was.

Dr. Barnett sought to detect the effect of the electric intensity arising from the uniform rectilinear motion of a magnet. My contention was that, according to recognized electromagnetic theory, the case surrounding his detecting condenser shielded the latter completely from the effects of the electric intensity. In other words, the electric intensity in question resulted in an induced charge-distribution on the surface of the case; and, the portion of the resultant electric intensity inside the case, due to this induced charge, completely cancelled the other portion, whose effect was the primary object of the investigation. This conclusion rested upon the fact that, in Dr. Barnett's experiment, the electric intensity in question was of a type derivable from a potential. I had read Dr. Barnett's papers to which he refers, and have reëxamined them since, but have failed to find any reference to this matter, which constitutes the whole point of my paper, and which indeed appears to me to render obvious, from the start, the null result to be expected from the experiment.

I may add that my desire was to show that Dr. Barnett's result was entirely consistent with electromagnetic theory as expressed by the Maxwell-Lorentz equations, a point which seemed to me worthy of mention because Dr. Barnett considers his experiment inconsistent with the theory of relativity with which, at any rate in its correctly stated form, all conclusions properly derived from electromagnetic theory must be consistent.

UNIVERSITY OF MINNESOTA.

[1] PHYS. REV., June, 1920, pp.527–528.

[2] PHYS. REV., March, 1920, p. 227. I have given a further account of the point concerned, in a paper covering the whole subject of Unipolar Induction, and published in PHYS. REV., V. 15, pp. 365–398, 1920.

[3] PHYS. REV., August, 1918, pp. 95–114.

The Electrostatic Charges of the Earth and Sun and their Relation to Terrestrial Magnetism.

By Fernando Sanford.

UP to the present time no one has proposed any explanation of the earth's magnetic field except the possible rotation of a negative charge with the earth.

If the magnetic fields of the sun and earth are due to the rotation of their negative charges, these charges should be great enough to exert an inductive influence upon each other, and the earth should be less electronegative upon its day side than upon its night side. The existence of this phenomenon has been shown by a series of photographic records of the diurnal changes in the earth's potential at Palo Alto, California, for the months of August, September and October and up to this writing on November 16.

These records have been made by means of a quadrant electrometer, one pair of whose quadrants is connected to the water system of Palo Alto while the other pair is joined to an insulated capacity enclosed in an earthed metal cage.

The average diurnal change in potential for the period mentioned has been about 70 millivolts, as registered by the electrometer. What relation this bears to the actual change in the potential of the earth is not known. The maximum negative potential has occurred at about 8 P.M., while the maximum positive potential which occurred between 9 and 10 A.M., in August has changed progressively to a later period until it occurred at noon in October. If the potential at 6 A.M., be taken as the zero potential for the day then the earth's potential is negative throughout the night and positive throughout the day. This is shown graphically by Fig. 1, which represents the diurnal potential curve for 29 days from August 6 to September 4, 1920.

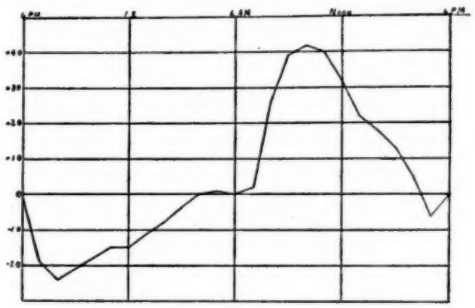


Fig. 1.

Since the magnetic field induced by the rotation of the earth's charge should be parallel to its axis of rotation, the magnetic variation produced by a change in the earth's potential should appear most plainly in the variation of the north-south component of the magnetic field for the same period. Since the electric charge which is assumed to produce the earth's magnetic field must be

negative, the N magnetic component should be greater where the earth is more negative, and *vice versa*. The curve for the N magnetic component should then resemble the electric potential curve with its signs inverted.

The only data which I have been able to find for the diurnal variation of the N magnetic component for the month of August are published by Chree in Volume II. of the Collected Researches of the National Physical Laboratory, of Teddington, England. These data are taken from the records of the Falmouth Observatory for the five most quiet days of each August for the twelve years, 1891–1902. They are represented graphically in the dotted curve in Fig. 2, the continuous curve being the electric potential curve at Palo Alto, as shown by Fig. 1, except that the mean potential of the day is

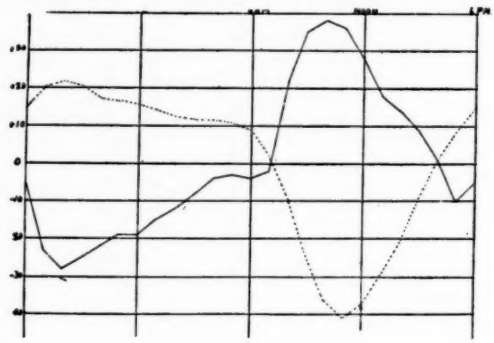


Fig. 2.

taken as zero, instead of the potential at 6 A.M. The relation between the two curves, representing, as they do, conditions at places one third the circumference of the earth distant in longitude, 14 degrees distant in latitude and 20 years distant in time, is so close as to preclude any possibility of their not being related to the same physical cause.

The same relation is shown equally well by the corresponding curves for September and October, an interesting fact being that the time of greatest deviation in both sets of curves has moved progressively from about ten o'clock in August to Noon in October.

A further semblance to terrestrial magnetism is shown in the disturbances which resemble "magnetic storms." Fig. 3 shows photographic copies of

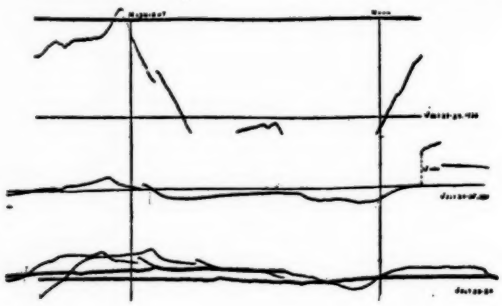


Fig. 3.

three daily records during the course of one of these disturbances. The break in the middle record marked 1.5 volt shows the effect of introducing a dry cell of this voltage between the electrometer and the earth. The lower record shows parts of two days. The heavy line in this record is a magnetometer record showing change in declination.

Fig. 4 shows copies of three successive daily records from August 3 to 6.

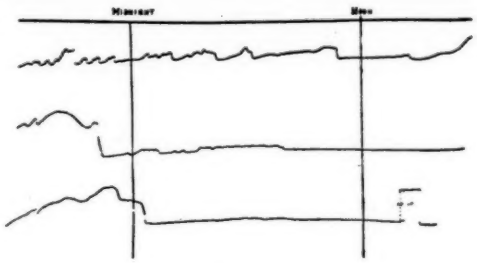


Fig. 4.

In these records the disturbances give the suggestion of interfering wave-trains. The repetition of the "bay" in the two later records shows a common characteristic of magnetic storms. The effect of introducing 1.5 volts between the electrometer and the earth is shown in the lower record. The original records are 20 inches long.

PALO ALTO, CALIFORNIA,
November 16, 1920.

THE SPECTROPHOTOELECTRIC SENSITIVITY OF PROUSTITE.

BY W. W. COBLENTZ.

Continuing the investigation of the spectrophotoelectrical properties of substances, an examination was made of the mineral proustite, Ag_3AsS_3, using the apparatus described in previous papers.

The samples examined were beautiful semitransparent crystals of a light vermilion color, indicating a high absorptivity for wave-lengths less than 0.55 μ.

At $+20°$ to $-50°$ the spectrophotoelectric sensitivity curve of proustite has a slight maxim at about 0.61 μ and a marked sensitivity with a maximum in the extreme violet. As the temperature is lowered (to $-100°$ C.) the maximim in the ultra-violet is more and more obliterated by a new maximum (the 0.61 μ band) which occurs at about 0.58 μ. The position of this new maximum remains quite constant as the temperature is decreased to $-170°$ C. No photoelectric sensitivity was observed for radiation stimuli of wave-lengths extending from 1 μ to 2 μ in the infra-red.

The general behavior of the spectrophotoelectric reaction in proustite is quite similar to that of cuprous oxide;[1] and it is entirely different from silver

[1] Pfund, PHYS. REV., (2) 7, p. 289; 1916.

sulphide,[1] which has the maximum of its photoelectrical sensitivity at 1.2 μ in the infra-red.

BUREAU OF STANDARDS,
WASHINGTON, D. C.,
November 10, 1920.

COLOR FILTERS FOR PHOTOGRAPHIC USES.

BY CHARLES D. HODGMAN.

IN the course of a somewhat extended study of the processes of three-color photography a considerable number of filters of various types have been worked out. They include a series of three-color filters of varied extent and abruptness of absorption, various yellow or orange contrast or compensation filters and others for various special uses such as photomicrography and contrast control in the photography of colored objects.

The filters constructed were of the ordinary type, each consisting of two glass plates flowed with colored gelatine and, when dry, cemented together with balsam. The factors determining the selective absorption are the strength and dye content of the gelatine solution and the volume flowed per unit area. The flowing solution consisted in each case of six per cent. by weight of clarified gelatine, a definite quantity of one or more aqueous dye solutions and distilled water to make the whole up to 100 per cent. Any filter is thus fully determined by giving for each of its two components (1) the dye used and the strength of aqueous stock solution, (2) the quantity of this dye solution used in 100 parts of the final mixture, and (3) the quantity flowed on unit area. The gelatine solution is filtered and poured while still warm on the previously cleaned and leveled glass plates.

The action of the filters is shown by photographs of their absorption spectra. It is possible however to indicate the characteristics approximately by giving the limits of action in the spectrum. Such an attempt is made in the accompanying table in connection with sufficient data for exact reproduction.

In choosing filters for any specific purpose it should be noted that the action obtained is limited by the sensitiveness of the photographic plate used and the character of the light source. Due allowance must be made for these factors. The following data refers to absorption spectra for which the light source was an incandescent tungsten filament in a gas filled bulb of light blue color. This lamp is supposed to produce an approximation to daylight and is commercially known as "Mazda C 2." The spectrum of this source on the brand of panchromatic plate used is continuous from about 0.35 μ to 0.72 μ. A region of less action exists at about 0.52 μ. The ordinary plates, not color sensitized, showed action without filter from 0.35 μ to 0.55 μ. In the case of the orthochromatic plates the action was extended to about 0.63 μ.

The first group listed is a series of red filters differing principally in the position of the limit of absorption toward the blue. While originally intended

[1] Coblentz & Kahler, B. S. Bulletin, 15, p. 213; 1919.

Filter.	Formula.			Plate.	Limits of Action, Microns.
	Dye Solution.	Quantity.	Coating.		
		c.c.	c.c.		
110	*a* Rhodamine B 1%	85	0.094	Pan.	.60–.71
	b Orange G 1%*	50	.125		
103	*a* Rose Bengal 0.5%	13.6	.094	Pan.	.58–.71
	b Orange G 1%*	50	.125		
12	*a* Erythrosine 0.5%	20	.094	Pan.	.57–.71
	b Orange G 1%*	50	.125		
5	*a* Ponceau 2R 1%	15	.094	Pan.	.56–.71
	b Auramine 0.5%	20	.094		
4	*a* Ponceau 2R 1%	12	.094	Pan.	.55–.71
	b Orange G 1%*	20	.094		
115	*a* Fluoresceïne 0.5%*	94	.156	Pan.	.54–.71
	b Picric acid 0.5%	33	.094		
113	*a* Ponceau 2R 1%	5	.094	Pan.	.54–.71
	b Metanil yellow 0.5%	16	.125		
13	*a* Naphthol green 0.5%	12	.094	Pan.	.48–.62
	Brilliant green 0.5%	2			
	b Picric acid 0.5%	30	.125		
	Naphthol yellow sat. sol.	50			
8	*a* Naphthol green 0.5%	6	.094	Pan.	.48–.62
	Brilliant green 0.5%	3			
	b Picric acid 0.5%	94	.125		
7	*a* Methylene green 1%	15	.125	Pan.	.47–.62, slight action
	b Auramine 0.5%	20	.094		.66–.71
121	*a* Picric acid 0.5%	30	.125	Ortho.	.46–.63
	Naphthol yellow sat. sol.	50			
	b Uncoated glass				
17	*a* Naphthol yellow sat. sol.	50	.125	Ortho.	.47–.63
	b Naphthol yellow sat. sol.	50	.125		
14	*a* Nacht blau 1%	15	.094	Pan.	.40–.49
	b Gentian violet 0.5%	10	.094		
10	*a* Methylene green 0.5%	15	.125	Pan.	.37–.49
	b Gentian violet 0.5%	20	.094		
11	*a* Iodine green 1%	10	.094	Pan.	.35–.51
	b Gentian violet 0.5%	10	.078		
119	*a* Picric acid 0.5%	5	.078	Ord.	.39–.50
	b Erythrosine 0.5%	20	.094		
123	*a* Aesculin 1%	5	.094	Ord.	.39–.50
	b Erythrosine 0.5%	15	.094		
118	*a* Aesculin 1%	5	.094	Ord.	.39–.52
	b Uncoated glass				
20	*a* Orange G 1%*	23	.094	Pan.	.53–.70
	b Naphthol yellow sat. sol.	50	.094		
19	*a* Orange G 1%*	8	.094	Pan.	.52–.70
	b Naphthol yellow sat. sol.	50	.094		
17	*a* Naphthol yellow sat. sol.	50	.125	Pan.	.49–.70
	b Naphthol yellow sat. sol.	50	.125		

Filter.	Formula. Dye Solution.	Quantity.	Coating.	Plate.	Limits of Action, Microns.
		c.c.	c.c.		
121	*a* Picric acid 0.5% + Naphthol yellow sat. sol.	30 + 50	.125	Pan.	.47–.70
	b Uncoated glass				
122	*a* Aesculin 1%	15	.125	Pan.	.42–.70
	b Naphthol yellow sat. sol.	50	.078		
118	*a* Aesculin 1%	5	.094	Pan.	.39–.70
	b Uncoated glass				
16	*a* Naphthol yellow sat. sol.	50	.062	Pan.	.35–.70, max. .56
	b Uncoated glass				
18	*a* Orange G 1%*	1.5	.094	Pan.	.40–.70, max. .56
	b Naphthol yellow sat. sol.	50	.062		
23	*a* Gentian violet 0.5%	10	.094	Pan.	.64–.71
	b Orange G 1%*	50	.125		
101	*a* Naphthol green 0.5% + Orange G 1%*	20 + 50	.125	Pan.	.59–.65
	b Erythrosine 0.5%	94	.125		
104	*a* Fluoresceïne 0.5%*	94	.125	Ortho.	.54–.61
	b Picric acid 0.5%	33	.094		
20	*a* Orange G 1%*	23	.094	Ortho.	.52–.61
	b Naphthol yellow sat. sol.	50	.094		
17	*a* Naphthol yellow sat. sol.	50	.125	Ord.	.47–.53
	b Naphthol yellow sat. sol.	50	.125		
107	*a* Nacht blau 1%	20	.094	Pan.	.46–.51
	b Picric acid 0.5%	33	.078		
106	*a* Rose Bengal 0.5%	16	.125	Ord.	.46–.50
	b Picric acid 0.5%	33	.078		
111	*a* Picric acid 0.5% + Erythrosine 0.5%	33 + 20	.094	Pan.	.46–.48
	b Nacht blau 1%	20	.094		
108	*a* Eosine yellowish 1%	14	.094	Ord.	.42–.47
	b Aesculin 1%	15	.125		
22	*a* Eosine yellowish 1%	14	.094	Ord.	.37–.45
	b Eosine yellowish 1%	14	.094		
106	*a* Rose Bengal 0.5%	16	.125	Pan.	.46–.50, .59–.71
	b Picric acid 0.5%	33	.078		
123	*a* Erythrosine 0.5%	15	.094	Pan.	.39–.51, .55–.71
	b Aesculin 1%	5	.094		
119	*a* Picric acid 0.5%	5	.078	Pan.	.39–.48, .56–.71
	b Erythrosine 0.5%	20	.094		
108	*a* Eosine yellowish 1%	14	.094	Pan.	.42–.47, .56–.71
	b Aesculin 1%	15	.125		
22	*a* Eosine yellowish 1%	14	.094	Pan.	.38–.45, .56–.71
	b Eosine yellowish 1%	14	.094		

*Orange G and fluoresceïne will not dissolve in water in the proportion stated. Ammonium hydroxide is added drop by drop until solution occurs.

for three-color work there are many other uses for which such filters are suitable.

The next two groups comprise green and blue three-color filters of various characteristics. Among these are certain yellow and negative green filters giving with ordinary or orthochromatic plates a range of action desirable in the blue and green sensation negatives in three-color work.

The series of orange and yellow contrast filters is in effect a continuation of the red series, absorbing decreasing amounts of the blue and violet. Two, Nos. 16 and 18 show an approach to the distribution of visual intensity.

The fifth group comprise filters which of themselves or when combined with the limited sensitiveness of ordinary or orthochromatic plates restrict the action to relatively narrow limits.

The last five filters, previously listed for use with ordinary or orthochromatic plates, are here used as negative green filters with panchromatic plates.

Characteristics and Formulæ of Filters.

For each component part of each filter, indicated as "*a*" and "*b*," there are given (1) the name of the dye used and the concentration of the aqueous stock solution, (2) the quantity of stock dye solution used for each 100 c.c. of the final mixture and (3) the volume of coating solution for each sq. cm. area. In addition to the dye solutions each coating mixture contains six per cent. by weight of gelatine and distilled water to make 100 per cent.

CASE SCHOOL OF APPLIED SCIENCE.

COMPARISON BETWEEN THE FUNDAMENTAL EQUATIONS FOR THE PONDEROMOTIVE FORCE FOR POINT CHARGES DUE TO LARMOR-LORENTZ AND TO MEGH NAD SAHA.

BY ALBERT C. CREHORE.

THE following five points are deduced from the form of the expressions for the ponderomotive forces due to Larmor-Lorentz and to Megh Nad Saha for point charges. They may each be taken as in favor of the probable truth of the Saha form in preference to the Larmor-Lorentz form. The equations under discussion are

$$\mathbf{F} = e_1\left(\mathbf{E} + \frac{1}{c}\mathbf{q}_1 \times \mathbf{H}\right), \tag{1}$$

where

$$\mathbf{E} = \frac{e_2(1-\beta_2^2)}{(RA_2)^3}\left\{\mathbf{R} - \frac{R}{c}\mathbf{q}_2 + \frac{\left\{\mathbf{f}_2 \times \left(R - \frac{R}{c}\mathbf{q}_2\right)\right\} \times \mathbf{R}}{c^2(1-\beta_2^2)}\right\}, \tag{2}$$

$$\mathbf{H} = \frac{e_2(1-\beta_2^2)}{(RA_2)^3}\left\{-\frac{\mathbf{R}\times\mathbf{q}_2}{c} + \frac{\left\{\mathbf{f}_2 \times \left(\mathbf{R} - \frac{R}{c}\mathbf{q}_2\right)\right\} \times \mathbf{R}}{Rc^2(1-\beta_2^2)}\right\}, \tag{3}$$

$$A_2 = 1 - \frac{\mathbf{q}_2\cdot\mathbf{R}}{cR} \quad \text{and} \quad A_1 = 1 - \frac{\mathbf{q}_1\cdot\mathbf{R}}{cR}, \tag{4}$$

The Saha equation is

$$\mathbf{F} = \frac{e_1 e_2 (1 - \beta_2^2)}{(RA_2)^3 (1 - \beta_1^2)^{1/2}} \left\{ \left(1 - \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{c^2} \right) \mathbf{R} - \frac{RA_1}{c} \mathbf{q}_2 \right\}. \tag{5}$$

1. The Larmor-Lorentz form (1) makes the force upon a moving point charge due to a stationary point charge equal to the simple electrostatic force. That is to say, the force does not depend upon the state of motion of the first charge.

Under the same conditions the Saha equation (5) makes the force depend upon the velocity of the first point. These statements are shown to be true by putting $\beta_2 = 0$ in the two sets of equations, when the Larmor-Lorentz form gives

$$\mathbf{F} = \frac{e_1 e_2}{R^3} \mathbf{R}, \tag{6}$$

and the Saha form gives

$$\mathbf{F} = \frac{e_1 e_2}{R^3 (1 - \beta_1^2)^{1/2}} \mathbf{R}. \tag{7}$$

A criticism of the Larmor-Lorentz equation on these grounds was given on pages 465–6, PHYS. REV., June, 1917. The Saha equation has supplied this deficiency.

2. The force upon the first point charge due to the second in the Larmor-Lorentz form obeys the inverse first power of the distance, when the distance is great. In the Saha form it obeys the inverse square of the distance law. This may be shown from the equations. In the expressions (2) and (3), the last term or the vector containing the acceleration, $\left[\mathbf{f}_2 \times \left(\mathbf{R} - \frac{R}{c} \mathbf{q}_2 \right) \right] \times \mathbf{R}$, contributes a term varying as R^2, and this divided by the R^3 in the coefficient gives a result R^{-1}. Now, any term varying as R^{-1} may be made larger than one varying as R^{-2} or higher powers by taking the distance large enough.

On the other hand the Saha equation contains no acceleration term, and no term varying as R^{-1}, the lowest order being R^{-2}. Again, these inverse first power terms in the Larmor-Lorentz form may be picked out from the equation when the electron has a circular motion. The first term of the force (1), namely $e_1\mathbf{E}$, is given in full by equations (48), (49) and (50), page 453 PHYS. REV., June, 1917. If we suppose that the first point is stationary $\mathbf{q}_1 = 0$, and the magnetic term in (1) vanishes. The whole force at great distances exerted upon a stationary charge by a revolving charge is, therefore, according to the Larmor-Lorentz form, neglecting the inverse square of the distance and higher powers as inappreciably small in comparison,

$$\mathbf{F}_i = \frac{e_1 e_2 \beta_2^2}{R^2 A_2^3 a_2} \left\{ -\beta_2 y \cos\alpha + \frac{1}{R} [(z^2 \cos\alpha - xz \sin\alpha) S_2 + y^2 S_2 \cos\alpha - xy C_2] \right\} i, \tag{8}$$

$$\mathbf{F}_j = \frac{e_1e_2\beta_2^2}{R^2A_2^3a_2}\left\{\beta_2(x\cos\alpha + z\sin\alpha) + \frac{1}{R}[-xyS_2\cos\alpha + x^2C_2 + z^2C_2 - yzS_2\sin\alpha]\right\}j, \quad (9)$$

$$\mathbf{F}_k = \frac{e_1e_2\beta_2^2}{R^2A_2^3a_2}\left\{-\beta_2 y\sin\alpha + \frac{1}{R}[-yzC_2 + y^2S_2\sin\alpha - (xz\cos\alpha - x^2\sin\alpha)S_2]\right\}k. \quad (10)$$

For the definition of terms reference may be made to the paper cited.

3. As to the forces exerted by one body upon another, let it be assumed that each body is made up of neutral atoms, each atom containing as much charge that is stationary as it does charges that revolve. When the Larmor-Lorentz equation is applied to such bodies, and it is considered that the action of any pair of charges is unaffected by the presence of all the others, and a summation taken, the result is that the total force is zero, that is in the line joining centers. It does not matter whether the negative or the positive charges are considered as the rotators.

On the other hand the Saha equation applied to the two bodies under the same conditions gives a residual attraction between the neutral bodies. Moreover, this attraction obeys the Newtonian law of equal action and reaction, the force being inversely as the square of the distance. It is shown under another title that this result is directly traceable to the extra factor $(1 - \beta_1^2)^{-1/2}$ in the Saha equation (5).

4. The tangential force upon one selected electron in a ring of equally spaced electrons is in the direction of rotation of the ring in the Larmor-Lorentz form. The force referred to is that due to the other electrons in the ring. This fact has probably been responsible for the use of rings in an atomic model more than any other circumstance. But, suspicions might well have been more vigorously aroused as to the correctness of this result, because the equation has never led to an exact solution of an equilibrium speed, where the velocity is uniform. There is always a radiation of energy demanded, and its source of supply must have been the internal energy of the electrons, which must, therefore have been changing.

The Saha equation leads to a diametrically opposite result. The force along the tangent line has the opposite direction to that deduced from the Larmor-Lorentz equation, and tends to stop the ring.

Being willing to follow the lead of the new ideas suggested by the Saha equation, the author has come to the conception of an atom without revolving rings of electrons, but with a central positive nucleus revolving, a Copernican system instead of a Ptolemaic system. This topic is discussed under another title.

5. The Saha ponderomotive force equation possesses much greater simplicity than that of Larmor-Lorentz. There are but two instead of four directions in the component vectors composing the force as expressed by (5), namely that

of the line joining centers and of a direction opposite to the velocity of the second point. From a mathematical viewpoint the Saha equation possesses the advantage of perfect symmetry in each of the four components of the force resolved along the four axes of the four-dimensional space. With a mathematician this point possibly outweighs all of the others mentioned.

NELA RESEARCH LABORATORIES.

THE NEWTONIAN LAWS OF GRAVITATION DEDUCED FROM THE SAHA ELECTROMAGNETIC THEORY APPLIED TO THE COPERNICAN ATOM.

BY ALBERT C. CREHORE.

IN a previous paper[1] the Larmor-Lorentz form of ponderomotive force was applied to rings of electrons at a great distance apart, and the author was led to the conclusion that the fundamental ponderomotive force equation requires modification. The evidence for this seemed so strong that he deliberately modified the resulting expression but not the fundamental equation, obtaining for the force exerted upon a first ring by a second ring

$$F = \frac{E_1 E_2}{2k} \beta_1^2 \beta_1^2 \{1 - (-X \sin \alpha + Z \cos \alpha)^2\} r^{-2}, \qquad (1)$$

X and Z being the direction cosines of position of the center of the second ring with respect to the first, α the angle between their axes of revolution, and k the specific inductive capacity of the medium. E_1 and E_2 represent the total charges in the rings, and β_1 and β_2 their respective speeds relative to the velocity of light.

When the directions of the axes of the rings are oriented on the supposition that all directions are equally probable, the brace in (1) becomes $2/3$, thus reducing the expression to

$$F = \frac{E_1 E_2}{3k} \beta_1^2 \beta_2^2 r^{-2}. \qquad (2)$$

The new ponderomotive force equation due to Saha leads to precisely this equation (1) as the force exerted upon a first neutral atom due to a second neutral atom, assuming that (1) is summed for each pair of rings in each of two atoms. Using the Saha equation the average force for all orientations exerted by a second ring upon a first ring is

$$F = \frac{E_1 E_2}{k} [-1 + (\tfrac{2}{3}\beta_2^2 - \tfrac{1}{2}\beta_1^2) + (\tfrac{3}{20}\beta_2^4 + \tfrac{1}{3}\beta_1^2\beta_2^2 - \tfrac{3}{8}\beta_1^4) \cdots] r^{-2}, \qquad (3)$$

where E_1 and E_2 represent the total charge per ring. Using the Larmor-Lorentz equation this force is

$$F = \frac{E_1 E_2}{k} [-1 - \tfrac{2}{3}\beta_2^4 - \tfrac{4}{5}\beta_2^6 - \cdots] r^{-2}. \qquad (4)$$

[1] PHYS. REV., June, 1917, p. 445. "The Atom," D. Van Nostrand Co., New York, p. 118, equation (201).

The important difference to be noticed is that the Saha equation contains a term with the product of the two speeds, β_1^2, and β_2^2, whereas the Larmor-Lorentz equation does not. The result is that, when all the forces between two bodies composed each of neutral atoms is summed up, the whole force vanishes at great distances for an equation like (4), while for a form like (3) the product terms will remain and represent the whole force. To show this let us write by menas of (3) the total force exerted by all revolving charges in body B upon all revolving charges in A. We have

$$F_1 = \frac{1}{k}[-\Sigma E_1\Sigma E_2 + \tfrac{2}{3}\Sigma E_1\Sigma(E_2\beta_2^2) - \tfrac{1}{2}\Sigma(E_1\beta_1^2)\Sigma E_2 + \tfrac{3}{20}\Sigma E_1\Sigma(E_2\beta_2^2) + \tfrac{1}{3}\Sigma(E_1\beta_1^2)\Sigma(E_2\beta_2^2) - \tfrac{3}{8}\Sigma(E_1\beta_1^4)\Sigma E_2]r^{-2}. \quad (5)$$

The force due to the revolving charges in B upon all the stationary charges in A is obtained by putting $\beta_1 = 0$ and changing the sign of (5), giving

$$F_2 = \frac{1}{k}[\Sigma E_1\Sigma E_2 - \tfrac{2}{3}\Sigma E_1\Sigma(E_2\beta_2^2) - \tfrac{3}{20}\Sigma E_1\Sigma(E_2\beta_2^4)]r^{-2}. \quad (6)$$

The force due to the stationary charges in B upon the revolving charges in A is obtained from (5) by making $\beta_2 = 0$, and changing the sign, giving

$$F_3 = \frac{1}{k}[\Sigma E_1\Sigma E_2 + \tfrac{1}{2}\Sigma(E_1\beta_1^2)\Sigma E_2 + \tfrac{3}{8}\Sigma(E_1\beta_1^4)\Sigma E_2]r^{-2}. \quad (7)$$

The force due to the stationary charges in B upon the stationary charges in A is obtained from (5) by making both β_1 and β_2 equal to zero, and not changing the sign, giving

$$F_4 = -\frac{1}{k}\Sigma E_1\Sigma E_2 r^{-2}. \quad (8)$$

The total force of B on A is the sum of these four parts. (5) to (8), giving

$$F = \frac{1}{3k}\Sigma(E_1\beta_1^2)\Sigma(E_2\beta_2^2)r^{-2}. \quad (9)$$

This is precisely the same as (2) above when E_1 and E_2 stand for integral numbers of electrons, and assuming that the force expressed by (2) is summed over the two bodies A and B. Moreover, the form would have come out the same as (1), had we started with the equation from which (3) has been derived that is, before the average for orientation was taken.

It has been shown[1] that the force expressed by (2) or by (9) or by (1) is an exact statement of Newton's laws of gravitation provided we concede that all rotators in a crystal may be divided into four equal groups with axes of rotation parallel respectively to the four medial lines of a regular tetrahedron. With this arrangement of the axes of rotation the force becomes independent of the orientation of the crystal. The force is always an attraction, and obeys the inverse square law. When, however, the values of the speeds which electrons were supposed to have in rings are substituted in this equation, the force comes out of the order 10^{27} times too large to be the gravitational force.

[1] "The Atom," Chapter XIII. and following.

But, having obtained such a result from the Saha equation, it is well to give heed to some other matters that this equation reveals. Chief among these is the fact that the force upon a single electron in a ring due to all the other electrons in the ring is in the direction opposite to its motion instead of in the same direction as the Larmor-Lorentz equation requires. In following the Saha theory I have, therefore, abandoned the employment of rings of electrons as a feature of an atomic model. Previous to this investigation the author had already concluded that the nucleus of an atom must revolve. The Lorentz mass formula applied to a positive charge of $2e$, uniformly distributed as in the solid Lorentz electron, gives for its radius

$$ak = \frac{4}{5m}\left(\frac{2e}{c}\right)^2 = \frac{16}{5m}\left(\frac{e}{c}\right)^2. \tag{10}$$

The reciprocal of e^2/m_Hc^2 is twice the Rydberg constant, $2K$, namely

$$2K = m_H\left(\frac{c}{e}\right)^2. \tag{11}$$

If this is a true physical equation, and $2e$ is the charge on the nucleus of the hydrogen atom, then by (10) and (11)

$$a_Hk = 8/5K. \tag{12}$$

These relations are satisfied dimensionally by giving the specific inductive capacity, k, the dimensions $L^{-1}T$, the reciprocal of a velocity. But, (12) connects the radius of the nucleus, a, immediately with the Rydberg constant, and shows that there ought to exist within the nucleus the characteristic property of a frequency, K. The value $2K$ is the frequency that the author had formerly used as the frequency of revolution of the electrons in hydrogen, and it also agrees with the Bohr value of the frequency of revolution of his single electron.

By changing over from the Ptolemaic or ring system to the Copernican, or rotating nucleus system for an atomic model, and retaining the same frequency of rotation, $2K$, for the nucleus as before used for the negative electron in an orbit, the magnitude of the gravitational force given by (9) is found exact without any modification. Moreover, when the positive nucleus is set in rotation it becomes equivalent to a closed ring of point charges, so far as great distances are concerned, and a ring even according to the Larmor-Lorentz theory is known to lose no energy through radiation. The two greatest difficulties with an atomic model are thus at once overcome.

The comparison between the magnitudes of the actual and theoretical attractions, therefore, reduces to a comparison between the actual radius of the nucleus of an atom and the theoretical radius of a ring which gives the same attraction. Let us make the comparison using two hydrogen atoms assuming that the nucleus has a positive charge $2e$. Its radius by the Lorentz formula (10) is then $a = 4.86 \times 10^{-16}$ cm. The Newtonian attraction on the average between two hydrogen atoms is

$$F = k'm_H^2r^{-2}. \tag{13}$$

The theoretical attraction is according to (9)

$$F = \frac{e^2}{3K}\beta^4 r^{-2}. \tag{14}$$

Equating gives

$$\beta^4 = 3kk'\left(\frac{m_H}{e}\right)^2; \qquad \beta^2 = \sqrt{3kk'}\,\frac{m_H}{e}. \tag{15}$$

With the numerical values, $k = 1$; $k' = 666. \times 10^{-10}$; $m_H = 1.658 \times 10^{-24}$ and $e = 4.763 \times 10^{-10}$, we find

$$\beta^2 = 0.778 \times 10^{-18} \qquad \text{and} \qquad \beta = 0.882 \times 10^{-9}. \tag{16}$$

Hence the linear velocity is $v = 26.5$ cm. per sec. approximately. Using the frequency of rotation, $2K$, as above stated, we obtain

$$v = 4\pi aK, \qquad \text{and} \qquad a = \frac{v}{4\pi K} = 6.4 \times 10^{-16} \text{ cm.} \tag{17}$$

The best that can be done at present is to compare this radius with 4.86×10^{-16} cm., that obtained from the mass formula on the assumption of no rotation. The value obtained for the equivalent ring is about one third larger. This, however, is considered to be in remarkable agreement in view of the fact that a charge set in rotation has not been investigated theoretically. There is little doubt in the author's opinion that such a charge set in rotation changes its shape, possibly assuming the form of an oblate spheroid, or some other more complicated form. At any rate, up to a certain limit, any change in shape would seem likely to bring the two figures just given into closer accord.

The two radii would have been $4\sqrt{2} = 5.66$ times further from agreement had we assumed that the nucleus of the hydrogen atom has a single charge of e, and used the same frequency.

It is scarcely necessary to point out that this Copernican atom possesses features that are likely to prove of considerable interest to chemists, as well as to those who have seen advantages in the so-called magneton, for the nucleus becomes in effect a magneton.

NELA RESEARCH LABORATORIES.

ON THE PROPAGATION OF THE SOUND WAVE FROM THE MUZZLE OF A LARGE GUN.

BY DAYTON C. MILLER.

IN 1918 and 1919 an extended series of experiments was carried out at Sandy Hook Proving Ground, on the various characteristics—pressure, velocity, and wave form—of the air waves generated by large guns in action. One group of observations, consisting of seventy sets of measures, relates to the velocity of propagation of the wave. This problem has three phases: (1) The velocity of an explosive sound of great intensity; (2) the normal velocity of sound in free air, and (3) the form of the wave front and its propagation from the muzzle of the gun. The calculations relating to the last group of observations have

now been completed, including corrections for the meteorological conditions.

The observations were made, by means of a microphone and string-galvanometer method, along the axis at intervals of 100 feet up to 1,000 feet from the gun, and also along lines making angles of about 45°, 90°, and 180° with the axis. In other experiments observations were made to distances as great as 2,200 feet from the gun. The observations have been reduced by means of a least squares solution, and are completely satisfied by the following construction for the wave front.

The trace of the expanding wave front on a plane containing the line of fire is always a circle. The center of the circle moves forward from the muzzle of the gun along the axis with a velocity, the initial value of which depends upon the kind and size of gun and the charge of powder. The velocity of displacement of the center diminishes rapidly, being an exponential function of the time. The radius of the circle at any given time after the origin of the wave is the distance sound would travel in air in this time with the uniform normal velocity of sound for the given meteorological conditions. In other words, the wave front develops in such a manner that if the distance which the sound is considered to have traveled is measured from the virtual instantaneous center, instead of from the muzzle of the gun, the velocity of the wave front is always the normal velocity of sound. The equation of the wave front when the muzzle of the gun is taken as the origin, is then, that of a circle with a moving center, as follows:

$$[x - a(1 - e^{-bt})]^2 + y^2 = v^2t^2,$$

a being the maximum displacement of the center along the axis, b the damping coefficient of the air for this displacement, t the time elapsed since the origin of the sound at the muzzle of the gun, and v the normal velocity of sound in air under the given meteorological conditions. For the complete determination of the wave front it is only necessary to determine experimentally the propagation along the axis for the given gun and charge of powder. The position of the wave front on the line of fire at a given time t is, then

$$x_t = vt + a(1 - e^{-bt}),$$

and the velocity is,

$$\frac{dx_t}{dv} = v + abe^{-bt},$$

while the initial velocity when $t = 0$, is $v + ab$.

For a 10-inch rifle, firing the service charge, the total displacement of the center, a, has been found to be about 72 feet, which value is attained in about 0.2 second after the discharge of the gun, and the damping coefficient, b, has the value 27.

Corrections for temperature and humidity are applied in the usual manner. If there is a wind, since the medium moves as a whole, the wave-front remains a circle, but the center is displaced in accordance with the velocity of the wind.

These equations have been found to satisfy all the observations very closely, the largest single residual corresponding to 0.4 foot in the position of a station,

which is equivalent to 0.0004 second in time, up to distances of about a thousand feet.

CASE SCHOOL OF APPLIED SCIENCE,
November 20, 1920.

A NEW TONE GENERATOR.

BY C. W. HEWLETT.

THE instrument consists of two equal pancake coils of insulated wire placed close together, with a thin aluminum diaphragm in between. Each coil is built up of annular coils with annular air spaces between to allow sound waves to pass out from either side of the diaphragm. The coils and diaphragm are held in place by insulating materials so that magnetic hysteresis is eliminated, and eddy currents cut down to a minimum. This is one of the chief features of the instrument as it enables it to be used at frequencies where the ordinary telephone receiver would fail, on account of the tremendous losses owing to hysteresis and eddy currents.

To operate the instrument a direct current battery is connected to the two coils in series in such a way that the north face of one coil faces the north face of the other. The resulting magnetic field in the region occupied by the diaphragm lies in the direction of the radii of the diaphragm and may be referred to as the "polarizing" field of the instrument. Two fairly large condensers are connected in series and this is connected to the two terminals of the battery. If now an alternating E.M.F. is impressed on this circuit at the common junction of the condensers, and the common junction of the two coils, the diaphragm will vibrate with the frequency of the alternating current. The reason is as follows: The alternating current passes through the coils so that at every instant the adjacent faces of the two coils so far as the alternating fields in the two coils are concerned, are of opposite sign. The resulting alternating magnetic field in the region of the diaphragm is normal to the diaphragm, and consequently this alternating magnetic field induces alternating currents in the diaphragm which are circular, and concentric with the center of the diaphragm. These alternating currents in the diaphragm react with the polarizing radial field and cause the diaphragm to vibrate with the same frequency and character as the alternating current. This instrument is most conveniently used in connection with a vacuum tube oscillator. By shunting the circuit as described above with a choke coil of very high impedance for the frequencies to be used, it may be included as part of the tuned portion of a vacuum tube oscillator circuit. By simply changing the tuning of this circuit the frequency of the sound produced may be varied over a wide range. The whole audible limit can be covered with this instrument.

The operation of the instrument has been investigated mathematically and it appears that it should be possible to use the instrument as a precision source of sound. On account of the elimination of magnetic hysteresis, and the absence of eddy currents except in the diaphragm where the loss due to this

cause can readily be calculated in any given case; it is possible to calculate the periodic force acting on the diaphragm at all places, for any given frequency; and from this to calculate the energy of virbation delivered to the air. Experiments are in progress along this line to determine the minimum intensity of sound for audition as a function of the frequency.

The instrument has also been used successfully as a transmitter and receiver of voice currents. The instrument should be particularly useful in the transmission and reception of speech on account of the elimination of magnetic hysteresis and eddy current losses which distort the voice currents in other types of apparatus.

STATE UNIVERSITY OF IOWA,
November 6, 1920.

A METHOD OF STUDYING SOUND WAVES BY MEANS OF A SYNCHRONOUS COMMUTATOR.

BY L. O. GRONDAHL.

IN a study of transmission of sound through the human body that was undertaken at the suggestion of and with the coöperation of Dr. William Charles White of the Tuberculosis League of Pittsburgh, there was developed a method of studying sound waves which seems worth while making known as it may have a number of applications in other sound problems. As it was necessary for the writer to leave the work in the initial stage, no applications have been carried out.

The method was designed to determine the relative transmission of sound from the open mouth through the lungs to different portions of the surface of the chest. The sound producer was in this case a telephone receiver excited by a current from a 900-cycle airplane generator. This diaphragm was held over the open mouth and the sound transmitted was received in a stethoscope bell held against the chest. The stethoscope bell was attached by means of a rubber tube to a specially designed microphone, kindly loaned for the purpose by the Western Electric Company. The current variations produced in the microphone circuit were received by the primary of a transformer, the secondary of which was connected to a two-stage amplifier. From the amplifier the alternating current was then brought back to a crown commutator mounted on the shaft of the alternator. The pulsating direct current thus produced was read on a D'Arsonval galvanometer.

It is evident that the deflection obtained depends on the setting of the brushes on the commutator. If commutation takes place at the current minimum, the galvanometer deflection is a maximum, and if it takes place at the current maximum, galvanometer deflection is zero. At successive current minima, the galvanometer deflection is alternately positive and negative. An illustration of this was immediately encountered in application to our problem when it was found that the reading with the stethoscope bell over the trachea was the negative of that with the bell over the lobe of the lung.

It is always possible to set the brushes on the commutator so as to get zero deflection on the galvanometer. That enables one therefore to determine the exact phase relation between the sound received at one position and that received at any other position in the medium. It makes it possible to map a sound wave front and to measure wave-lengths. If the source is stationary and the stethoscope bell is movable, one can choose a position of zero deflection in the galvanometer and follow it around this source. This might be interesting in such a problem as that of Abbott & Clarke on the velocity of sound from a moving source, or in a study of the effect of attaching a horn to a source of sound.

It can be shown that the method is independent of overtones and other sounds of a frequency different from the one that is being investigated.

If a number of commutators are mounted on the same shaft it makes it possible to analyze the sound that is used for the experiment. Commutators on which the number of segments are in the ratio of 1 : 2 : 3 : 4 etc. can be used to give readings corresponding to the fundamental, and to any overtone for which there is a corresponding commutator. It will also give the phase relation between the different harmonics. The fact that it is possible to separate out the components in this way makes it possible to study each one entirely independently of the rest. In a study of the relative intensities of components of different frequencies, it would, of course, be necessary to make a careful study of the characteristics of all the vibrating parts.

The work then suggests the following:

1. A D'Arsonval galvanometer method of comparing sound intensities that is independent of all sounds of frequencies that are different from the one that is being studied, and useful in the study of sound transmission.
2. A method of mapping sound waves in air.
3. A method of studying wave-lengths and sound velocities in a medium without depending on standing waves.
4. A method of analyzing the sound that is used in an experiment where the sound is produced directly or indirectly by a rotating mechanism.

Carnegie Institute of Technology,
September, 1920.

On the Relative Positions of Lines in X-Ray Spectra.

By William Duane and R. A. Patterson.

MANY phenomena indicate that the lines in the L series of X-rays divide themselves into at least three groups. The lines in any one group are associated together in some way. For instance they may be produced by the same subatomic mechanism.

Three critical absorption wave-lengths have been found in the L series of each chemical element (of high atomic number), and these belong respectively to the three corresponding groups of emission lines.

It has been shown by experiments[1] that in the K series the critical absorption

[1] Phys. Rev., Oct. 1919, p. 369 and Proceedings of the Nat. Ac., Sept., 1920.

has a wave-length a fraction of one per cent. shorter than that of the γ emission line, and no emission lines shorter than the γ line have been observed in the K series of any chemical element.

Our experiments on the spectrum of tungsten indicate that the third critical absorption in the L series, La_3, has a wave-length slightly shorter than that of the shortest L emission line, $L\gamma_4$. With regard to the position of La_3, therefore, the L series has the same structure as the K series.

When we examine the positions of the other two L critical absorptions, however, relative to the emission lines in their groups, we find a somewhat different situation. The first L group, L_1, contains the lines l, α_2, α_1, β_2 and β_5, β_5 having the shortest wave-length. In order to determine definitely the exact position of the first L critical absorption, La_1, with regard to these emission lines we performed the experiment in such a way as to measure the wave-lengths of La_1 and of the emission lines in a single series of readings. The graph shown on the slide[1] represents the electric current in the X-ray spectrometer's ionization chamber as a function of the grazing angle of incidence (and, therefore, approximately of the wave-length, λ). The sharp drop in this graph which corresponds to the critical absorption La_1 lies between the peaks that represent the two emission lines β_2 and β_5. La_1, therefore, must have a longer wave-length than β_5.

The second group of lines in the L series, to which the second critical absorption La_2 belongs, contains among others the emission line γ_1 and probably the emission line γ_2. The graph shown in the second slide[2] indicates the relative position of these lines. The sharp drop corresponding to La_2 lies between the peaks representing the lines γ_1 and γ_2. The line γ_2 has a shorter wave-length than the line γ_1, and, therefore, a shorter wave-length than the critical absorption La_2.

We may arrive at a theoretical explanation of the fact that a line may have a shorter wave-length than that of the critical absorption belonging to its group by assuming that the critical absorption corresponds to the transfer of an electron from an L orbit to an orbit at the periphery of the atom, where it can find a vacant place. This does not mean a transfer to an indefinitely great distance from the atom. According to this idea a line of shorter wave-length than the critical absorption would be due to an electron falling from an orbit outside of the periphery.

This conception does not contradict the quantum theory, provided that the critical potential required to produce such a line of short wave-length slightly exceeds that required to produce the other lines in the group, and that the frequency required to produce such a line in X-ray fluorescence slightly exceeds that required to produce the other lines.

There are other more or less plausible explanations.

The positions of the critical ionization wave-lengths in the L series are now being investigated.

HARVARD UNIVERSITY.

[1] See Proceedings of the Nat. Acad., Sept., 1920, p. 521.
[2] See Proceedings of the Nat. Acad., Sept., 1920, p. 522.

X-RAY ANALYSIS OF THREE SERIES OF ALLOYS.

BY MARY R. ANDREWS.

USING the Hull method of X-ray analysis the three series of alloys, nickel-iron, cobalt iron and copper-zinc were examined.

The nickel-iron alloys show centered cubic (iron) structure from 0 to 20 per cent. Ni, a mixture of centered cubic and face-centered cubic (Ni) to about 35 per cent. Ni, and face-centered cubic alone from 35 per cent. to 100 per cent. Ni.

A non-magnetic nickel-iron showed almost pure face-centered cubic structure in the non-magnetic condition, but strongly centered-cubic form after magnetization by chilling.

The cobalt-iron alloys show the centered cubic (iron) form from 0 to 75 per cent. Co, the face-centered cubic from 85 to 98 per cent. Pure cobalt crystallizes in both face-centered cubic and in hexagonal forms.

The zinc copper alloys show three structures: 0 to 40 per cent Zn, face-centered cubic (copper form); 40 to 50 per cent. Zn, face-centered and centered cubic mixture; 50 to 70 per cent. Zn, nearly pure centered cubic; 70 to 80 per cent. Zn—mixture of centered cubic and another form (probably rhombohedral); 80 to 90 per cent. Zn—this form nearly pure; 90 to 95 per cent. Zn mixture of this form and hexagonal (Zn form).

RESEARCH LABORATORIES,
GENERAL ELECTRIC COMPANY.

THE RELATION BETWEEN THE EMISSIVE POWER OF A METAL AND ITS ELECTRICAL RESISTIVITY.

BY C. DAVISSON AND J. R. WEEKS.

MAXWELL'S electromagnetic theory of light, Kirchoff's radiation law and Planck's law of power distribution in the black body spectrum together constitute sufficient basis for calculating in detail the characteristics of the thermal radiation from a metal. The only special data required are the temperature of the metal and its electrical resistivity.

On this basis Foote has derived an expression for the total emissive power of a metal defined in terms of normal intensity of radiation, and has shown that emissive powers calculated from this expression agree very exactly with observed values in the case of platinum.

It does not seem to be generally recognized, however, that Foote's formula is not appropriate for the calculation of total emissive powers defined in terms of power radiated per unit area, or, which amounts to the same thing, that the departures from Lambert's cosine law in the infra-red are by no means negligible.

The difference to be expected between these two differently defined emissive powers may be arrived at from the following considerations. The coefficient of reflection of a metal of resistivity r (ohm-cm.) for unpolarized radiation of wave-length λ (cm.) meeting its surface at angle of incidence Φ turns out, on

Maxwell's theory, to be

$$R = \frac{\sqrt{F(\Phi)} - \sqrt{2}\cos\Phi}{\sqrt{F(\Phi)} + \sqrt{2}\cos\Phi} \cdot \frac{G(\Phi)}{G(\Phi) + \sqrt{2}\sin\Phi\tan\Phi\, F(\Phi)},$$

where

$$F(\Phi) = (3600\lambda^2/r^2 + \cos^4\Phi)^{1/2} + \cos^2\Phi$$

and

$$G(\Phi) = F(\Phi) - \cos^2\Phi + \sin^2\Phi\tan^2\Phi.$$

The emissive power of the metal for this wave-length and angle is $E = (1 - R)$ and the total power radiated per unit area at temperature T is

$$W_T = 2\int_0^\infty J_\lambda \int_0^{\pi/2} (1 - R)\sin\Phi\cos\Phi\, d\Phi\, d\lambda,$$

where J_λ is Planck's distribution function.

The total emissive power of the metal defined in terms of total power radiated is

$$E = \frac{2\int_0^\infty J_\lambda \int_0^{\pi/2} (1 - R)\sin\Phi\cos\Phi\, d\Phi\, d\lambda}{\int_0^\infty J_\lambda d\lambda}.$$

On the other hand the total emissive power defined in terms of normal intensities is

$$E' = \frac{\int_0^\infty J_\lambda(1 - R_0)d\lambda}{\int_0^\infty J_\lambda d\lambda},$$

where R_0 is the value of R for $\Phi = 0$.

One has, therefore, for the ratio of the two emissive powers

$$\frac{E}{E'} = \frac{\int_0^\infty J_\lambda 2\int_0^{\pi/2} (1 - R)\sin\Phi\cos\Phi\, d\Phi}{\int_0^\infty J_\lambda(1 - R_0)d\lambda}.$$

Figure 1 shows the ratio

$$\frac{2\int_0^{\pi/2} (1 - R)\sin\Phi\cos\Phi\, d\Phi\, d\lambda}{(1 - R_0)}$$

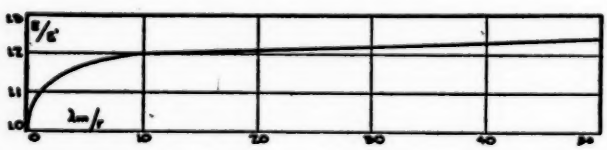


Fig. 1.

as a function of λ/r. It will be noticed that this ratio changes only slowly with λ except for λ/r small. For this reason the ratio E/E' is approximately equal to the above ratio calculated for $\lambda/r = \lambda m/r$ where λ_m has its usual meaning, and is given by $\lambda_m = (0.2940/T)$.

The total emissive power of a metal defined in terms of total power radiated may, therefore, be calculated by first calculating its emissive power defined in terms of normal intensities by Foote's formula,

$$E' = 0.5736(rT)^{1/2} - 0.1769(rT)$$

and then multiplying the value of E' by the ratio E/E' taken from Fig. 1 and corresponding to $\lambda m/r = (0.2940/(rT))$.

Theoretical and observed results for platinum are shown in Fig. 2. The curve r gives the experimentally determined resistivity-temperature relation

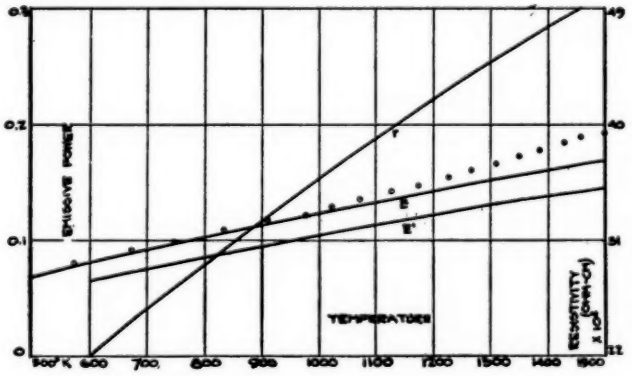


Fig. 2.

for a sample of pure platinum wire. E' is the emissive power curve for this sample calculated from Foote's formula. E is the curve of emissive power defined in terms of total power radiated and obtained from the curve E' as described above. The circles indicate values of emissive power determined by dividing the power dissipated in vacuo by a uniformly heated length of the wire by the power radiated by an equal area of black body at the same temperatures. The points are, therefore, experimentally determined values corresponding to curve E. It will be noticed that the agreement between observed and calculated values is good up to about 1000° K. and that above this temperature the observed values are above the calculated curve. This departure is taken to be due to the variation of resistivity with frequency which becomes important as the bulk of the radiation moves toward short wave-lengths.

It appears, therefore, from both theoretical considerations and from observation that very great departures from Lambert's cosine law occur in the infrared, resulting in the case of platinum in differences between the two differently defined total emissive powers of as much as 20 per cent.

RESEARCH LABORATORIES OF THE AMERICAN TELEPHONE AND TELEGRAPH COMPANY AND THE WESTERN ELECTRIC COMPANY, INCORPORATED, NEW YORK CITY,

MASS-ABSORPTION COEFFICIENTS AS A FUNCTION OF WAVE-LENGTH ABOVE AND BELOW THE K X-RAY LIMIT OF THE ABSORBER.

BY F. K. RICHTMYER.[1]

IN a paper[2] read at the April (1920) meeting of the Physical Society data was presented showing the relation between X-ray wave-lengths and the mass-absorption coefficient μ/ρ for wave-lengths from 0.1 Å. up to 0.5 Å. μ/ρ was shown to be a linear function

$$\left(\frac{\mu}{\rho} = A\lambda^3 + \frac{\sigma}{\rho}\right)$$

of the cube of the wave-length for the three substances investigated; aluminum, copper, and molybdenum. The present paper is an extension of this work to longer wave-lengths, and to two substances of higher atomic number, silver and lead. In the case of molybdenum, silver, and lead the law of absorption has been studied on the long wave-length side of the K absorption limit.

Aluminum.—The previous equation: $\mu/\rho = 14.45\,\lambda^3 + .15$ was found to give values of μ/ρ slightly too high for wave-lengths above 0.5 Å., there being a slight systematic variation above this point. The equation

$$\frac{\mu}{\rho} = 14.3\,\lambda^3 + .16$$

gives μ/ρ within about 1 per cent. up to 0.8 Å., but does not fit quite so well as the original equation at the shorter wave-lengths.

Molybdenum.—Values of μ/ρ have been obtained up to $\lambda = .85$ Å., thus passing through the K absorption limit at $\lambda = .622$ Å. Below this limit a graph between μ/ρ and λ^3 is very slightly concave toward the λ^3 axis, the original equation

$$\frac{\mu}{\rho} = 450\,\lambda^3 + .4$$

giving values, very exact up to .3 Å. but deviating increasingly above that value.

As is well known the absorption takes an abrupt drop at the gamma line of the K series, but even on the long wave-length side of the gamma line the linear relation between μ/ρ and λ^3 still holds with different value of the coefficient of λ^3 but, so far as data can show, *with the same values of* σ/ρ. Above the gamma line the equation is

$$\frac{\mu}{\rho} = 51.5\,\lambda^3 + .5.$$

Silver.—Values of μ/ρ have been measured up to $\lambda = .7$ Å. Below the K absorption limit the equation

$$\frac{\mu}{\rho} = 603\,\lambda^3 + .7$$

represents absorption quite accurately, especially at the shorter wave-lengths.

[1] This work was done at the Research Laboratory of the General Electric Co., Schenectady, to which the writer wishes to make grateful acknowledgment.

[2] PHYSICAL REVIEW, Vol. XV. (1920), p. 547.

Above the K limit the values of μ/ρ are given by

$$\frac{\mu}{\rho} = 86\,\lambda^3 + .6.$$

Lead.—Only the region on the long wave-length side of the K absorption limit has been meausred. The equation

$$\frac{\mu}{\rho} = 510\,\lambda^3 + 1$$

fits the data reasonably well from .17 Å. up to .7 Å. It is to be pointed out that in the region from the K limit of lead (λ = .148 Å.) up to the K limit of silver (λ = .491 Å.) silver is a better absorber than lead.

In the previous paper it was shown that A, in the equation

$$\frac{\mu}{\rho} = A\lambda^3 + \frac{\sigma}{\rho}$$

is proportional to the cube of the atomic number of the absorbing material for values of λ less than the critical K-absorption limit, using data on Al (N = 13), Cu (N = 29), and Mo (N = 42). This has been now extended to include Ag (N = 47). It is to be pointed out, however, that for each substance there seems to be a slight deviation from this law as the K_γ line is approached, the values of μ/ρ being increasingly less than would be predicted from this equation.

If we represent the absorption above the K-series limit by

$$\frac{\mu}{\rho} = L\lambda^3 + \frac{\sigma}{\rho},$$

σ/ρ, as noted above, seems to have the same numerical value as below that limit, while the ratio of A to L is of the order of 7 : 1. Data herein contained is not sufficiently accurate to determine whether L is proportional to the cube of the atomic number. The observations seem to indicate a power somewhat higher than the cube.

From the above therefore it seems possible to represent the mass absorption coefficient of any substance by an equation of the form

$$\frac{\mu}{\rho} = [K\lambda^3]_0^K + [L\lambda^3]_0^L + \frac{\sigma}{\rho},$$

where $K = A - L$. The term $[K\lambda^3]_0^K$ is to be used from $\lambda = 0$ to $\lambda = K_\gamma$, the absorption limit of the K series. The term $[L\lambda^3]_0^L$ is to be used from $\lambda = 0$ up to the first absorption limit of the L series, beyond which L is probably split up into components. If radiation of wave-length $\lambda < K_\gamma$ falls on an absorber the first of these two terms determines the proportion of total energy reradiated as K radiation. The second determines the proportion reradiated as L, M (etc.) radiation.

In conclusion, the simplicity of absorption laws in the X-ray region as compared with the corresponding laws (whatever they may be!) in the visible and ultra-violet region point to the desirability of approaching the latter by gradually extending our knowledge of the former to longer and longer wave-lengths.

CORNELL UNIVERSITY,
October, 1920.

GRAPHICAL DETERMINATION OF HEXAGONAL AND TETRAGONAL CRYSTAL STRUCTURE FROM X-RAY DATA.

BY ALBERT W. HULL AND WHEELER P. DAVEY.

X-RAY diffraction patterns from powdered crystals may be interpreted very easily in case the crystals are cubic. In case the crystals belong to any other system, the interpretation has, until now, been easy only in case correct quantitative crystallographic data could be found. The method here reported makes the tretagonal and hexagonal systems as easy of interpretation as the cubic.

The spacings of all possible planes (within a predetermined range) are plotted[1] as a function of the axial ratio. These spacings are plotted on a logarithmic scale (shown on the bottom of each plot) so that they may be compared with the experimental values without regard to the absolute values of the spacings.

To use one of the plots, a straight-edged piece of paper is placed beneath the logarithmic scale of abscissas and the experimental values of planar spacings are laid off on the edge. This paper strip is then moved over the plot with the edge always parallel to the axis of abscissas until a position is found where an exact match is obtained. When this is accomplished the correct crystal system and type of lattice has been found, viz.: that for which the plot was calculated; the axial-ratio is shown by the intercept on the axis of ordinates; and the crystal form corresponding to each experimental spacing may be read off at the top of the plot.

Three plots are given for the hexagonal system, and three for the tetragonal, as follows:

Hexagonal System.

(1) Single triangular prism lattice.
(2) Two triangular prism lattices in close packed arrangement.
(3) Three triangular prism lattices in rhombohedral arrangement.

Tetragonal System.

(1) Single tetragonal lattice.
(2) Body centered tetragonal lattice.
(3) Face centered tetragonal lattice.

The rhombohedral and tetragonal plots may also be used for cubic crystals, and the axial ratios at which these may be read off are indicated on the graphs.

The use of the plots is illustrated by the analysis of the diffraction patterns of zinc, cadmium, and indium. The arrangement of atoms in zinc is shown to be that of hexagonal close packing with axial-ratio 1.86, and side of unit triangular prism 2.670 Å.

The arrangement in cadmium is that of hexagonal close packing with axial ratio 1.89, and side of unit triangle 2.98 Å.

[1] These plots will be published in full in the PHYSICAL REVIEW.

The lattice of indium is face centered tetragonal with axial-ratio 1.06, and side of unit square 4.58 Å.

Crystallographic data for the last of these is lacking in the literature. For the first two, the axial ratios in the literature are incorrectly given as 1.356 and 1.335 respectively.

RESEARCH LABORATORY,
GENERAL ELECTRIC CO.,
SCHENECTADY, N. Y.,
November 17, 1920.

THE ABSORPTION AND SCATTERING COEFFICIENTS FOR HOMOGENEOUS X-RAYS IN SEVERAL ELEMENTS OF LOW ATOMIC WEIGHT.

BY C. W. HEWLETT.

THIS work was done during the summer of nineteen hundred twenty at the Research Laboratory of the General Electric Company in Schenectady, New York.

Homogeneous X-rays were secured by means of a Coolidge water cooled tube and rock salt spectrometer. X-rays from 0.10 to 1.05 Ångstrom wave-length were used. The total mass absorption coefficients were measured in water, aluminum, carbon, oxygen, nitrogen, lithium and iron. The absorption coefficients were found to be proportional to the cube of the wave-length with certain important exceptions. The mass scattering coefficients were all found to be less than that calculated for electrons 10^{-13} cm. in diameter, although Compton's[1] Theory of an electron 1.85×10^{-10} cm. diameter does not satisfactorily account for the facts. A complete discussion of the investigation is to be published very shortly.

STATE UNIVERSITY OF IOWA,
IOWA CITY, IOWA,
October 5, 1920.

THE CURRENT-TEMPERATURE RELATION FOR DIFFERENT PYROMETER FILAMENTS.

BY W. E. FORSYTHE.

IN the disappearing-filament type of optical pyrometer the calibration curve consists of the relation between the temperature of the standard source used and the current through the pyrometer filament. For convenience in reading, everything else being equal, the change in current for a particular change in temperature should be as large as possible. It has often been stated that carbon pryometer filaments were better than tungsten filaments because carbon at low temperatures gave a more open scale, that is, a greater change in current for a given change in temperature.

The relation between the pyrometer current and the corresponding tempera-

[1] A. H. Compton, PHYSICAL REVIEW, 14, 20–43, 247–259, 1919.

ture was determined for a range of temperatures from 1828° K. to about 950° K. for a number of tungsten and carbon filaments. For very short tungsten filaments where the effect due to the conduction of heat away from the filament by the ends reached to the central part of the filament the current-temperature relation was found to be much smaller than when the filaments were longer. The same effect also is true for short carbon filaments. However, if the carbon or tungsten filaments were long enough so that the end cooling did not effect the temperature of the central portion of the filament, there was not much difference in the ratio.

The ratio of the current range for a hairpin tungsten filament, length of loop 12 cm. was 4.2 for a temperature range from 1828° to 950° K., the ratio for a carbon hairpin loop 10 cm. long was 4.5 for the same temperature range. Both filaments were long enough so that the end cooling could be neglected.

NELA RESEARCH LABORATORIES,
CLEVELAND, OHIO,
November, 1920.

ON THE ABSORPTION SPECTRUM OF HCl.

BY WALTER F. COLBY AND CHARLES F. MEYER.

BECAUSE of the theoretical interest involved, the authors have undertaken to add to existing experimental data regarding the absorption of HCl in the infra-red. The method of investigation is the same as that used by Imes,[1] except that a longer absorption chamber is used, and provision is made for heating the gas.

Observations have thus far been made, chiefly on the short wave-length side of the center of the "fundamental," *i.e.*, in the region 3.47 μ to 3.16 μ. No evidence has yet been found of a definite head to the absorption band, but it is not said that none exists.

A part of the observations were made with a slit width of .25 mm. (half of that used by Imes), as it was thought that further structure of the lines constituting the band might be revealed. But the results give a first indication that there is no structure beyond that shown by Imes' curves.

The wave-lengths of twenty lines (absorption maxima) on the short wave-length side of the center have been measured. The agreement with Imes' measurements of the first twelve lines is fairly close, but the remaining eight lines show a more rapid convergence than he had prophesied. The twenty lines are represented by the formula:

$$\nu = 28{,}869 + 208.7n - 3.52n^2.$$

The fact that the frequencies are still represented by a parabola with the same degree of exactness, in passing from twelve members to twenty members, is strong evidence that this law is a real law of spacing.

It is hoped to improve the sensibility of the apparatus and to make further studies of the absorption of HCl and other halogen acids.

UNIVERSITY OF MICHIGAN, November, 1920.

[1] Imes, Astrophys. Journ. 48, p. 125, 1918.

Flame Excitation of Luminescence.

By E. L. Nichols and D. T. Wilber.

THIS new type of excitation, which was discovered in the course of a recent study of luminescence at high temperatures, has the following characteristics:

1. Luminescence is produced by actual contact of certain zones of a hydrogen flame with the material to be excited.

2. The active region of the flame is the boundary between reduction and oxidation.

3. Rapid reduction and oxidation appear to be the essential features of this process of light production. This is inferred from the fact that the best known flame is the hydrogen flame, the flames of alcohol, ether, sulphur, carbon disulphide, etc., having been found inactive while, on the other hand, a full supply of free oxygen in the atmosphere surrounding the flame is likewise essential.

4. The reduction occurring at the surface of the body thus brought to fluorescence by the action of the flame is presumably preliminary to oxidation which is probably to be regarded as the direct cause of luminescence.

This view of the phenomenon is supported by the following facts:

5. The effect cannot be produced by heating in air or in hydrogen outside the flame.

6. Strong electrostatic fields, variously applied, fail to modify the effect, thus apparently excluding ionization as an essential feature.

7. *Photo-excitation, as an explanation, is excluded* since:

(*a*) Many of the most striking instances of flame excitation occur in substances not capable of being excited by light: *e.g.*, CaO, MgO, ZnO, SiO_2, ZrO_2, Al_2O_3, exceptionally pure zinc sulphide and calcium sulphide, etc.

(*b*) Many strongly photo-luminescent compounds like calcium tungstate and willemite are inactive under flame excitation.

(*c*) Where photo-luminescent substances are excited as in the case of Sidot blende and certain of the phosphorescent sulphides of Lenard and Klatt, flame excitation continues to temperatures far above those at which photo-luminescence ceases.

(8) *The effect is not of the nature of temperature radiation*:

(*a*) It is in many cases visible below the red heat.

(*b*) It does not follow Wien's law but exists only between perfectly definite temperature limits.

(*c*) The spectrum is a characteristic luminescence spectrum and not a temperature spectrum. It consists of the usual overlapping bands which in turn are made up of components forming series of equal frequency intervals.

(*d*) The afterglow, which is very brief is of the type of vanishing phosphorescence.

Physical Laboratory of Cornell University,
October, 1920.

A CONTINUOUS SPECTRUM FROM MERCURY VAPOR.

BY CLEMENT D. CHILD.

UNDER certain conditions the discharge through mercury vapor gives a green glow which shows the ordinary line spectrum of mercury together with one which is continuous from the red into the ultra-violet being very prominent in the green. This is the same spectrum as that shown by fluorescent mercury.

There are two conditions necessary for obtaining this by electrical discharge with any considerable brightness; the discharge must pass through hot mercury vapor that is not very far from the point of condensation and the current density must be small. As the current density is increased the line spectrum becomes more prominent and the continuous spectrum less so. With a condenser in the circuit only the line spectrum appears. Either alternating or direct current may be used. The glow at the cathode shows the line spectrum more prominently than does the positive column.

The presence of the continuous spectrum does not depend on the kind of glass, on the electrode used, nor on the purity of the mercury.

When the vapor is in the proper condition for giving the continuous spectrum it differs from the vapor at room temperature in the following respects. It fluoresces under the action of ultra-violet radiation and as a result ionization is not necessary in order to produce visible radiation. It is ionized at a lower potential. The carriers of the continuous spectrum are not charged. The spectrum is radically different. The glow continues for a longer time after the excitation ceases.

The relative brightness of different parts of the spectrum is apparently the same under all conditions.

These facts can be explained by assuming that the carriers of the continuous spectrum are molecules of two or more atoms of mercury and that these molecules are more easily excited when the temperature is somewhat above that of room temperature.

The carrier of the continuous spectrum is probably excited in the case of electric discharge as in the case of fluorescent mercury by ultra-violet radiation of wave-length 2536. This radiation is no doubt excited by the impact of the electron on the mercury atom.

A continuous spectrum is also found in the light produced by the discharge through sodium vapor.

COLGATE UNIVERSITY,
October, 1920.

NOTE ON SHEET STRIÆ AS THIN FILMS IN OPTICAL GLASS.

BY L. E. DODD.

SHEET striæ in optical glass are very thin regions where the index differs slightly from that of the glass outside them. Extreme thinness and definiteness of bounding surfaces are indicated by observations such as Michelson's.[1] Samples have been obtained in the optical glass factory of the

[1] Michelson, Bureau of Standards Scientific Paper, No. 333, pp. 33 and 34.

Bureau of Standards, demonstrating the high tenacity, or cohesiveness, of these striæ, which enables them to maintain their sheet structure when subject to agents that deform them. Instead of piercing and passing through such a sheet, a bubble, or a foreign particle of solid matter of some size, will drag the sheet locally along with it, giving at first the appearance of say a rock suspended in a net in air, or a croquet ball in a hammock. As the bubble or solid particle moves farther along it continues to drag the stretching sheet with it, and locally transforms it to a stria of the tubular, or the hair, type.

Because of this definiteness of the bounding surfaces and the high cohesiveness the samples demonstrate, a sheet stria seems to be a true film in the main body of the glass, as a soap bubble is a thin film in air. The glass film is remarkable as its substance is supposedly very near physically and chemically to that of the medium in which it exists immersed. Surface tension forces would be expected under the circumstances to be much weaker. The present samples indicate clearly the striking individuality of the films.

Since in these glass films definite bounding surfaces and surface tension are indicated, the question arises as to the physical relation between the film surfaces and the surrounding mass of glass. Does the glass mass "wet" the film, like water in contact with dry wood, or does it act like a drop of mercury on the same surface? The magnitudes of the forces coming into play in the dragging of the film through the glass mass by either bubble or solid particle, depend upon which case holds. If there is no adhesion there must be more nearly pure slippage, with viscosity forces playing a somewhat different rôle than in the other event.

These striæ offer an interesting study in molecular mechanics. Rayleigh[1] describes some experiments having to do with slippage and non-slippage between solid surfaces in which it is shown that there can be a sticking between two such surfaces when perfectly cleaned, which a slight greasing destroys. It would be interesting to know the extent to which there is such a sticking between the glass film and the glass mass in the molten state, or whether there is present a constituent in such a state that it has a greasing effect, or whether there is present a thin layer giving the "incipient seizing" or whether there is entire absence of adhesion.

Finally, a proposed experiment is the production of a film, if not the blowing of a bubble, of suitable liquid material in a suitable liquid, to approximate the sheet striæ in optical glass.

Bureau of Standards,
Washington, D. C.,
November 6, 1920.

[1] Lord Rayleigh, "On the Lubricating and Other Properties of Thin Oily Films," Philosophical Magazine. 35. 157. 1918.

NEW BOOKS.

Advanced Lecture Notes on Light. By J. R. ECCLES. Cambridge: The University Press, 1919. Pp. 1 + 141.

The topics treated by the author are given in the following order: Rainbows, Magnifying Power, Chromatic and Spherical Aberration, Wave Theory, Interference, Diffraction, Polarization.

Some of these items are worked out in considerable detail with numerical examples. The volume is printed in quarto and on one side of the paper so that it can be used as an actual notebook.

Études de Photochemie. By VICTOR HENRI. Paris: Gauthier-Villars et Cie, 1919. Pp. vii + 218.

This is the first installment of an extended treatise covering ten years of work by the author and his colaborers.

The six chapters included in the present volume are devoted to absorption spectra of organic compounds; with special reference to the relations between absorption bands in the ultra-violet and the infra-red.

La Matérialisation de L'Énergie. By LOUIS ROUGIER. Paris: Gauthier-Villars et Cie, 1919. Pp. xii + 148.

A philosophical essay on Relativity and the Quantum Theory with special reference to what the author calls the *dematerialization* of matter and the *materialization* of energy.

Theorie der Strahlung und der Quanten. By DR. ARTHUR MARCH. Leipzig: Johann Ambrosius Barth, 1919. Pp. vii + 182. Price, M. 14.

This treatise on Radiation and Quanta, is in two nearly equal parts. The first deals with thermo-dynamic and electro-dynamic theories of radiation, including the introduction of the quantum concept. On this basis specific heats; spectral series, including the ultimate structure of line spectra; the Stark effect and entropy from the standpoint of the quantum theory are discussed.